Advanced Ruin Theory

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Spring 2023

Advanced Ruin Theory: generalities

- Scope-wise very close to earlier course *Queues and Lévy Fluctuation Theory.*
- Will be using lecture notes *The Cramér-Lundberg model and its variants — a queueing perspective* by M. Mandjes and O. Boxma.
- You have been sent *draft* of the book. Any comments are welcome, preferably by email. Publication in a few months.
- Twelve classes, roughly one per chapter.

Advanced Ruin Theory: practicalities

- Three homework sets. (Not in pairs.)
- Late May or in June: oral exam.
- Final grade average of the two individual grades.
- I'll use *Datanose* for sending out messages.

Advanced Ruin Theory: scope

- Risk process: models reserve level of insurance firm.
- Interested in probability of hitting 0: bankruptcy of insurance firm.
- Basic variant is *Cramér-Lundberg model*, but many (sophisticated) variants possible.
- Although we tell the story along the lines of risk theory, the material has a substantially broader applicability: extreme values of stochastic processes.
- Direct connection with queueing theory.
- In earlier course *Queues and Lévy Fluctuation Theory* we considered slightly broader class of processes, but derived slightly less explicit results.

CHAPTER I: CRAMÉR-LUNDBERG MODEL

Base model: *Cramér-Lundberg*

Setting considered:

- In CL model, clients of insurance firm generate independent and identically distributed (i.i.d.) claims, which arrive according to a Poisson process.
- Insurance firm receives premiums at constant rate.
- Key object: *ruin probability*, i.e., probability that for a given initial reserve, reserve level drops below zero.
- Two flavors: all-time ruin probability (ruin over an infinite time horizon) and time-dependent ruin probability (ruin before a given time).

Base model: *Cramér-Lundberg*

Duality with queueing:

- We often work with *net cumulative claim process*: cumulative amount of claimed money, decreased by the premiums earned.
- Insurance firm is ruined when net cumulative claim process exceeds the initial reserve.
- Consequence: ruin can be written in terms of the running maximum process (corresponding to net cumulative claim process) exceeding a given threshold (i.e., the initial reserve).
- Duality relation between event of ruin in CL model, and event of a workload threshold being exceeded in related M/G/1 queueing model.

Model description

- Claims arrive according to Poisson process with rate $\lambda > 0$. $N(t)$, number of claims in $[0, t]$, is Poisson with mean λt.
- Claims form sequence of i.i.d. random variables $B_1, B_2, \ldots$, distributed as generic non-negative random variable B with Laplace-Stieltjes transform (LST) given by

$$b(\alpha) := \mathbb{E}\, e^{-\alpha B} = \int_{[0,\infty)} e^{-\alpha t}\, \mathbb{P}(B \in dt).$$

- Clients generate premiums at constant rate $r > 0$.
- Initial reserve level is $u > 0$.

Until ruin, reserve level is given by (empty sum being defined as 0)

$$X_u(t) := u + rt - \sum_{i=1}^{N(t)} B_i.$$

Ruin probabilities

First objective: *all-time ruin probability*, for initial reserve level u, i.e., probability of $X_u(t)$ ever dropping below 0:

$$p(u) := \mathbb{P}(\exists s \geqslant 0 : X_u(s) \leqslant 0).$$

Second objective: *time-dependent ruin probability*, for initial reserve level u, i.e., probability of $X_u(t)$ dropping below 0 before t:

$$p(u,t) := \mathbb{P}(\exists s \in [0,t] : X_u(s) \leqslant 0).$$

Net-profit condition

- All-time ruin probability $p(u)$ is trivially 1 if *net-profit condition* $\lambda\,\mathbb{E}B < r$ is violated.
- Observe: $\lambda\,\mathbb{E}B$ is expected claimed amount per time unit, while r is the insurer's income per time unit.
- Time-dependent ruin probability $p(u, t)$ is worth studying regardless of whether or not net-profit condition holds.

Working with transforms

- Only in exceptional cases $p(u)$ and $p(u,t)$ allow explicit expression.
- Remedy: work with transforms, i.e.,

$$\pi(\alpha) := \int_0^\infty e^{-\alpha u} p(u)\, du.$$

- Time-dependent ruin: exponentially distributed time horizon ('killing'). Concretely, with T_β exponentially distributed time with mean β^{-1}, consider transform of $p(\cdot, T_\beta)$. Thus, focus on *double* transform

$$\pi(\alpha,\beta) := \int_0^\infty e^{-\alpha u} p(u, T_\beta)\, du = \int_0^\infty \int_0^\infty \beta\, e^{-\alpha u - \beta t} p(u,t)\, du\, dt.$$

- Abelian theorem: $\pi(\alpha) = \lim_{\beta \downarrow 0} \pi(\alpha,\beta)$. Hence: it suffices to focus on evaluating $\pi(\alpha,\beta)$ only.

Transform of running maximum

Define the 'net cumulative claim process' and corresponding running maximum process:

$$Y(t) := \sum_{i=1}^{N(t)} B_i - rt, \qquad \bar{Y}(t) := \sup_{s\in[0,t]} Y(s).$$

$Y(t)$: compound Poisson process with drift.

Clearly,

$$p(u) = \mathbb{P}(\bar{Y}(\infty) \geqslant u), \qquad p(u,t) = \mathbb{P}(\bar{Y}(t) \geqslant u).$$

Conclude: probabilities $p(u)$ and $p(u,t)$ are complementary cumulative distribution functions of random variables $\bar{Y}(\infty)$ and $\bar{Y}(t)$, respectively.

Transform of running maximum, ctd.

Consider

$$\varrho(\alpha,\beta) := \mathbb{E}\, e^{-\alpha \bar{Y}(T_\beta)} = \int_0^\infty e^{-\alpha u}\, \mathbb{P}(\bar{Y}(T_\beta) \in du).$$

Integration by parts:

$$\begin{aligned}\varrho(\alpha,\beta) &= -\int_0^\infty e^{-\alpha u}\, d\mathbb{P}(\bar{Y}(T_\beta) \geqslant u)\\ &= -e^{-\alpha u}\, \mathbb{P}(\bar{Y}(T_\beta) \geqslant u)\Big|_{u=0}^{\infty} - \alpha \int_0^\infty e^{-\alpha u}\, \mathbb{P}(\bar{Y}(T_\beta) \geqslant u)\, du\\ &= 1 - \alpha \int_0^\infty e^{-\alpha u}\, p(u, T_\beta)\, du = 1 - \alpha\pi(\alpha,\beta).\end{aligned}$$

Hence: when aiming at computing $\pi(\alpha,\beta)$, we can equivalently compute $\varrho(\alpha,\beta)$: these two double transforms uniquely define one another.

Duality with M/G/1 queue

- M/G/1 queue: reservoir at which i.i.d. jobs (distributed as a random variable B) arrive according to a Poisson process with rate $\lambda > 0$, drained at rate $r > 0$.
- $Q(t)$: workload in this system. Can be seen as net input process $Y(t)$ truncated at zero (thus preventing storage level from becoming negative). Assume $Q(0) = 0$.

Duality with M/G/1 queue, ctd.

Define the running minimum process by

$$\underline{Y}(t) := \inf_{s \in [0,t]} Y(s).$$

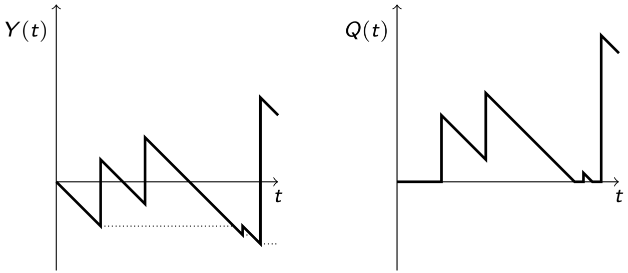

Figure: Net cumulative claim process $Y(t)$ (left panel) and workload process $Q(t)$ (right panel) for compound Poisson process. In left panel, corresponding running minimum process $\underline{Y}(t)$ is depicted by dotted lines.

Duality with M/G/1 queue, ctd.

From figure:

$$Q(t) = Y(t) - \underline{Y}(t).$$

In addition, relying on *time-reversibility* argument,

$$\begin{aligned} Y(t) - \underline{Y}(t) &= Y(t) - \inf_{s\in[0,t]} Y(s) = \sup_{s\in[0,t]} (Y(t) - Y(s)) \\ &\stackrel{d}{=} \sup_{s\in[0,t]} Y(s) = \bar{Y}(t), \end{aligned}$$

with '$\stackrel{d}{=}$' denoting equality in distribution.

Conclude: $\bar{Y}(t)$ has same distribution as $Q(t)$ ('duality').

Four methods to compute transform

- Method 1: use ruin model. Idea: condition on first event (either a claim arrival or having reached the time horizon).
- Method 2: use both ruin and queueing model. Idea: write running maximum as the sum of a geometric number of i.i.d. random quantities ('ladder heights').
- Method 3: use queueing model. Idea: rely on Kella-Whitt martingale and optional sampling machinery.
- Method 4: use queueing model. Idea: set up system of differential equations for the transform under study, and solve these.

Method 1: condition on first event

Roadmap:

- Evaluate $\pi(\alpha, \beta)$ by conditioning on first event, which is either a claim arrival or killing.
- Obtain an expression in terms of the transform of interest $\pi(\alpha, \beta)$.
- Solve $\pi(\alpha, \beta)$ from the resulting equation (also requiring identification of an unknown constant).

Method 1: condition on first event, ctd.

Recall: T_β is exponentially distributed with mean β^{-1}. Hence,

$$p(u, T_\beta) = \frac{\lambda}{\lambda+\beta}\Big(p_1(u, T_\beta) + p_2(u, T_\beta)\Big),$$

where, distinguishing between scenario that there is ruin due to first claim and scenario that multiple claims are needed,

$$p_1(u, T_\beta) := \int_0^\infty (\lambda+\beta)e^{-(\lambda+\beta)s}\int_{u+rs}^\infty \mathbb{P}(B\in dv)\, ds,$$

$$p_2(u, T_\beta) := \int_0^\infty (\lambda+\beta)e^{-(\lambda+\beta)s}\int_0^{u+rs} p(u+rs-v, T_\beta)\,\mathbb{P}(B\in dv)\, ds;$$

in latter expression, memoryless property of exponential distribution has been used.

Method 1: condition on first event, ctd.

We can thus write $\pi(\alpha,\beta) = \pi_1(\alpha,\beta) + \pi_2(\alpha,\beta)$, with

$$\pi_1(\alpha,\beta) := \int_0^\infty e^{-\alpha u} \int_0^\infty \lambda\, e^{-(\lambda+\beta)s} \int_{u+rs}^\infty \mathbb{P}(B \in dv)\, ds\, du,$$

$$\pi_2(\alpha,\beta) := \int_0^\infty e^{-\alpha u} \int_0^\infty \lambda\, e^{-(\lambda+\beta)s}$$

$$\int_0^{u+rs} p(u+rs-v,\, T_\beta)\, \mathbb{P}(B \in dv)\, ds\, du.$$

Next step: evaluate these by swapping order of integrals (and a change of variable).

Method 1: condition on first event, ctd.

Interchanging the order of the integrals,

$$\pi_1(\alpha,\beta) = \lambda \int_0^\infty \left(\int_0^v e^{-\alpha u} \left(\int_0^{(v-u)/r} e^{-(\lambda+\beta)s}\, ds \right) du \right) \mathbb{P}(B \in dv).$$

Then inner integrals can be evaluated:

$$\frac{\lambda}{\lambda+\beta} \int_0^\infty \left(\frac{1-e^{-\alpha v}}{\alpha} - \frac{e^{-(\lambda+\beta)v/r} - e^{-\alpha v}}{\alpha - (\lambda+\beta)/r} \right) \mathbb{P}(B \in dv).$$

This quantity can be interpreted in terms of the LST of B evaluated in specific values: with $s(\beta) := (\lambda+\beta)/r$,

$$\pi_1(\alpha,\beta) = \frac{\lambda}{\lambda+\beta} \left(\frac{1-b(\alpha)}{\alpha} - \frac{b(s(\beta)) - b(\alpha)}{\alpha - s(\beta)} \right).$$

Method 1: condition on first event, ctd.

Performing change of variable $w := u + rs$, $\pi_2(\alpha, \beta)$ equals

$$\frac{1}{r}\int_0^\infty e^{-\alpha u}\int_u^\infty \lambda\, e^{-s(\beta)(w-u)}\int_0^w p(w-v, T_\beta)\,\mathbb{P}(B\in dv)\,dw\,du.$$

Swap order of integrals:

$$\frac{\lambda}{r}\int_0^\infty \left(\int_v^\infty e^{-s(\beta)w}p(w-v, T_\beta)\left(\int_0^w e^{-\alpha u}e^{-s(\beta)(w-u)}\,du\right)dw\right)\mathbb{P}(B\in dv)$$

$$=\frac{\lambda}{r}\frac{1}{\alpha-s(\beta)}\int_0^\infty\left(\int_v^\infty\left(e^{-s(\beta)w}-e^{-\alpha w}\right)p(w-v, T_\beta)\,dw\right)\mathbb{P}(B\in dv).$$

But

$$\int_v^\infty e^{-\alpha w}p(w-v, T_\beta)\,dw = e^{-\alpha v}\int_0^\infty e^{-\alpha w}p(w, T_\beta)\,dw = e^{-\alpha v}\pi(\alpha,\beta),$$

(and likewise for $s(\beta)$ instead of α), so that

$$\pi_2(\alpha,\beta)=\frac{\lambda}{r}\frac{1}{s(\beta)-\alpha}\big(b(\alpha)\pi(\alpha,\beta)-b(s(\beta))\pi(s(\beta),\beta)\big).$$

Method 1: condition on first event, ctd.

- Add up expressions for $\pi_1(\alpha,\beta)$ and $\pi_2(\alpha,\beta)$.
- Observe that $\pi_2(\alpha,\beta)$ contains a term involving $\pi(\alpha,\beta)$.
- Solve for $\pi(\alpha,\beta)$.

Result:

$$\pi(\alpha,\beta) = r\frac{\lambda}{\lambda+\beta}\frac{s(\beta)-\alpha}{r(s(\beta)-\alpha)-\lambda b(\alpha)}\frac{1-b(\alpha)}{\alpha} - r\frac{\lambda}{\lambda+\beta}\frac{b(\alpha)-b(s(\beta))}{r(s(\beta)-\alpha)-\lambda b(\alpha)} - \frac{\lambda\, b(s(\beta))\,\pi(s(\beta),\beta)}{r(s(\beta)-\alpha)-\lambda b(\alpha)}.$$

Observe that right-hand side contains unknown quantity $\pi(s(\beta),\beta)$.

Method 1: condition on first event, ctd.

Constant $\pi(s(\beta), \beta)$ can be identified by using that a root of the denominator is also a root of the numerator.

Elementary: equation $r(s(\beta) - \alpha) - \lambda b(\alpha) = 0$ has for any $\beta > 0$ a unique positive root, say $\psi(\beta)$.

Leads to:

$$\begin{aligned}
\pi(s(\beta), \beta) &= \frac{r}{\lambda + \beta}\left(\frac{s(\beta) - \psi(\beta)}{b(s(\beta))}\frac{1 - b(\psi(\beta))}{\psi(\beta)} - \frac{b(\psi(\beta)) - b(s(\beta))}{b(s(\beta))}\right) \\
&= \frac{r}{\lambda + \beta}\left(\frac{s(\beta)(1 - b(\psi(\beta))) - \psi(\beta)(1 - b(s(\beta)))}{b(s(\beta))\,\psi(\beta)}\right).
\end{aligned}$$

Method 1: condition on first event, ctd.

Now define *Laplace exponent*

$$\varphi(\alpha) := \log \mathbb{E}\, e^{-\alpha Y(1)} = r\alpha - \lambda(1 - b(\alpha)).$$

Function $\psi(\cdot)$, as defined above, is inverse of Laplace exponent $\varphi(\cdot)$ (Check!) — in case $\varphi'(0) < 0$ actually *right* inverse.

Plugging in expression for $\pi(s(\beta), \beta)$ into $\pi(\alpha, \beta)$, after some calculus,

$$\begin{aligned}
\pi(\alpha, \beta) &= \frac{\lambda}{\varphi(\alpha) - \beta}\left(\frac{1 - b(\psi(\beta))}{\psi(\beta)} - \frac{1 - b(\alpha)}{\alpha}\right) \\
&= \frac{1}{\varphi(\alpha) - \beta}\left(\frac{\varphi(\alpha) - r\alpha}{\alpha} - \frac{\beta - r\psi(\beta)}{\psi(\beta)}\right) \\
&= \frac{1}{\varphi(\alpha) - \beta}\left(\frac{\varphi(\alpha)}{\alpha} - \frac{\beta}{\psi(\beta)}\right).
\end{aligned}$$

Method 1: condition on first event, ctd.

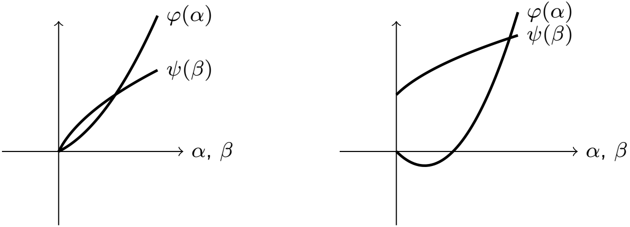

Figure: Functions $\varphi(\alpha)$ and $\psi(\beta)$ with $\varphi'(0) > 0$ (left panel) and with $\varphi'(0) < 0$ (right panel). In former case $\psi(0) = 0$, whereas in latter case $\psi(0) > 0$.

Method 1: condition on first event, ctd.

Now use result to derive expression for transform of $\bar{Y}(T_\beta)$, by translating $\varrho(\alpha,\beta)$ in terms of $\pi(\alpha,\beta)$.

Theorem (Time-dependent Pollaczek-Khinchine)

For any $\alpha \geqslant 0$ and $\beta > 0$,

$$\varrho(\alpha,\beta) = \frac{\alpha - \psi(\beta)}{\varphi(\alpha) - \beta}\,\frac{\beta}{\psi(\beta)}.$$

Exercise 1.1: procedure that uses this theorem to recursively evaluate all moments of running maximum $\bar{Y}(T_\beta)$.

Method 1: condition on first event, ctd.

Transform of $\bar{Y}(\infty)$ found by letting $\beta \downarrow 0$.
Net-profit condition needed, to make sure that $\bar{Y}(\infty)$ is finite.

Corollary (Pollaczek-Khinchine)

For any $\alpha \geqslant 0$, under the net-profit condition,

$$\varrho(\alpha) := \mathbb{E}\, e^{-\alpha \bar{Y}(\infty)} = \varrho(\alpha, 0) = \frac{\alpha\, \varphi'(0)}{\varphi(\alpha)}.$$

Method 1: condition on first event, ctd.

'Pollaczek-Khinchine' can be alternatively written as

$$\varrho(\alpha) = \frac{\alpha(r - \lambda\,\mathbb{E}B)}{r\alpha - \lambda(1 - b(\alpha))} = \left(1 - \frac{\lambda\,\mathbb{E}B}{r}\right) \Big/ \left(1 - \frac{\lambda}{r}\frac{1 - b(\alpha)}{\alpha}\right).$$

Observe: $(1 - b(\alpha))/(\alpha\mathbb{E}B)$ is transform of random variable $\bar{B}$ with density $f_{\bar{B}}(t) := \mathbb{P}(B \geqslant t)/\mathbb{E}B$:

$$\mathbb{E}\, e^{-\alpha\bar{B}} = \int_0^\infty e^{-\alpha u}\frac{\mathbb{P}(B \geqslant u)}{\mathbb{E}B}du = \frac{1 - b(\alpha)}{\alpha\,\mathbb{E}B}.$$

This implies $|(1 - b(\alpha))/(\alpha\,\mathbb{E}B)| \leqslant 1$ for $\alpha \geqslant 0$, so that we can write

$$\varrho(\alpha) = \left(1 - \frac{\lambda\,\mathbb{E}B}{r}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda\,\mathbb{E}B}{r}\right)^n\left(\frac{1 - b(\alpha)}{\alpha\,\mathbb{E}B}\right)^n.$$

Method 1: condition on first event, ctd.

Define $c := 1 - \lambda\,\mathbb{E}B/r \in (0,1)$, and let G be geometrically distributed with success probability c:

$$\mathbb{P}(G = n) = (1-c)^n c.$$

In addition, let $\bar{B}^{\star i}$ be a random variable defined as the sum of i i.i.d. copies of $\bar{B}$. Then we find the following representation.

Proposition (Geometric sum representation)

The following distributional equality applies: under the net-profit condition, an empty sum being defined as zero,

$$\bar{Y}(\infty) \stackrel{d}{=} \sum_{i=1}^{G} \bar{B}_i \stackrel{d}{=} \bar{B}^{\star G}.$$

Method 1: condition on first event, ctd.

Lemma

For any $\beta > 0$, $-\underline{Y}(T_\beta)$ is exponentially distributed with mean $1/\psi(\beta)$.

Proof. Process $K(t) := e^{-\varphi(\alpha)t}\, e^{-\alpha Y(t)}$ is a mean-1 martingale. Define $\sigma(v)$ as first time $Y(t)$ crosses level $-v$, for some given $v > 0$. Observe that $Y(\sigma(v)) = -v$ (Why?), so that by 'optional sampling'

$$1 = \mathbb{E}\, K(0) = \mathbb{E}\, K(\sigma(v)) = \mathbb{E}\left(e^{-\varphi(\alpha)\sigma(v)} 1\{\sigma(v) < \infty\}\right) \cdot e^{\alpha v}.$$

Plug in $\alpha = \psi(\beta)$:

$$\mathbb{E}\left(e^{-\beta\sigma(v)} 1\{\sigma(v) < \infty\}\right) = e^{-\psi(\beta)v}.$$

Stated follows from $\{-\underline{Y}(T_\beta) \geqslant v\} = \{\sigma(v) \leqslant T_\beta\}$ and Remark 1.3:

$$\mathbb{P}(-\underline{Y}(T_\beta) \geqslant v) = \mathbb{P}(\sigma(v) \leqslant T_\beta) = \mathbb{E}\left(e^{-\beta\sigma(v)} 1\{\sigma(v) < \infty\}\right) = e^{-\psi(\beta)v}.$$

Method 1: condition on first event, ctd.

- Note that

$$\mathbb{E}\, e^{-\alpha Y(T_\beta)} = \int_0^\infty \beta e^{-\beta t} e^{\varphi(\alpha) t} dt = \frac{\beta}{\beta - \varphi(\alpha)}.$$

- Time-reversal: $\bar{Y}(T_\beta) - Y(T_\beta) \stackrel{d}{=} -\underline{Y}(T_\beta)$. Due to Lemma, $\bar{Y}(T_\beta) - Y(T_\beta)$ is exponentially distributed with mean $1/\psi(\beta)$. So

$$\mathbb{E}\, e^{-\alpha(\bar{Y}(T_\beta) - Y(T_\beta))} = \frac{\psi(\beta)}{\psi(\beta) + \alpha}.$$

- By 'Time-dependent Pollaczek-Khinchine' and above results,

$$\begin{aligned} \mathbb{E}\, e^{-\alpha \bar{Y}(T_\beta)}\, \mathbb{E}\, e^{-\alpha(Y(T_\beta) - \bar{Y}(T_\beta))} &= \frac{\alpha - \psi(\beta)}{\varphi(\alpha) - \beta} \frac{\beta}{\psi(\beta)} \cdot \frac{\psi(\beta)}{\psi(\beta) - \alpha} \\ &= \frac{\beta}{\beta - \varphi(\alpha)} = \mathbb{E}\, e^{-\alpha Y(T_\beta)}. \end{aligned}$$

Method 1: condition on first event, ctd.

As (evidently)

$$Y(T_\beta) = \bar{Y}(T_\beta) + (Y(T_\beta) - \bar{Y}(T_\beta)),$$

above observations lead to the following result.

Proposition (Wiener-Hopf decomposition)

The random variables $\bar{Y}(T_\beta)$ and $\bar{Y}(T_\beta) - Y(T_\beta)$ are independent. The former has a Laplace-Stieltjes transform that is given by the time-dependent Pollaczek-Khinchine theorem, whereas the latter has the same distribution as $-\underline{Y}(T_\beta)$, i.e., is exponentially distributed with mean $1/\psi(\beta)$.

Method 2: ladder heights

Roadmap:

- $\bar{Y}(T_\beta)$ is distributed as sum of geometric number of i.i.d. copies of a ladder height H.
- Determine transform of H (also using some queueing-theoretic arguments).
- Determine $\varrho(\alpha, \beta)$.

Method 2: ladder heights, ctd.

Define $\tau_0 = 0$, and for $i = 1, 2, \ldots$,

$$\tau_i := \inf\left\{t \geqslant 0 : Y\left(t + \sum_{j=1}^{i-1} \tau_j\right) - Y\left(\sum_{j=1}^{i-1} \tau_j\right) > 0\right\},$$

$$H_i := Y\left(\sum_{j=1}^{i} \tau_j\right) - Y\left(\sum_{j=1}^{i-1} \tau_j\right).$$

- H_i: difference between the process' i-th and $(i-1)$-st record value;
- τ_i: time elapsed between epochs at which these two record values are attained.

$(H_i, \tau_i)_{i\in\mathbb{N}}$ is sequence of i.i.d. random vectors; let (H, τ) be corresponding generic random vector.

Method 2: ladder heights, ctd.

Busy period: uninterrupted interval in which associated queueing process is positive; are i.i.d., say distributed as generic random variable σ.

Observation: with B sampled independently of process $Y(t)$, busy period σ is distributed as first time $Y(t)$ crosses (stochastic) level $-B$.
Note: σ can be defective if net-profit condition is not fulfilled.

Method 2: ladder heights, ctd.

With $\sigma(x)$ as defined before,

$$\begin{aligned}\mathbb{E}\left(e^{-\beta\sigma}\,1\{\sigma<\infty\}\right) &= \int_0^\infty \mathbb{E}\left(e^{-\beta\sigma(x)}\,1\{\sigma(x)<\infty\}\right)\mathbb{P}(B\in dx)\\ &= \int_0^\infty e^{-\psi(\beta)x}\,\mathbb{P}(B\in dx) = b(\psi(\beta)).\end{aligned}$$

Using definition of $\varphi(\cdot)$, we find $\beta=\varphi(\psi(\beta))=r\psi(\beta)-\lambda(1-b(\psi(\beta)))$.

Lemma

For any $\beta>0$,

$$\mathbb{E}\left(e^{-\beta\sigma}\,1\{\sigma<\infty\}\right)=\frac{\beta+\lambda}{\lambda}-\frac{r}{\lambda}\psi(\beta).$$

Method 2: ladder heights, ctd.

Define

$$\begin{aligned}\xi(\alpha,\beta) &:= \mathbb{E}\Big(e^{-\alpha Y(T_\beta)}\,1\{Y(T_\beta)=\underline{Y}(T_\beta)\}\Big)\\ &= \mathbb{E}\Big(e^{-\alpha \underline{Y}(T_\beta)}\,1\{Y(T_\beta)=\underline{Y}(T_\beta)\}\Big).\end{aligned}$$

Proposition

For any $\alpha \geqslant 0$ *and* $\beta > 0$,

$$\xi(\alpha,\beta) = \frac{\beta}{r\psi(\beta) - r\alpha}.$$

Method 2: ladder heights, ctd.

Proof. $L(t) := -\underline{Y}(t)/r$ is the associated queue's idle time in $[0, t]$ (Check!). Hence, consider

$$\xi(\alpha, \beta) = \mathbb{E}\Big(e^{r\alpha L(T_\beta)}\, 1\{Q(T_\beta) = 0\}\Big).$$

Conditioning on the first event (killing time or start of a busy period), by exploiting the underlying regenerative structure,

$$\xi(\alpha, \beta) = \frac{\beta}{\lambda - r\alpha + \beta} + \frac{\lambda}{\lambda - r\alpha + \beta}\,\mathbb{P}(\sigma \leqslant T_\beta)\,\xi(\alpha, \beta).$$

Recalling that $\mathbb{P}(\sigma \leqslant T_\beta)$ can be rewritten as $\mathbb{E}\,(e^{-\beta\sigma}\,1\{\sigma < \infty\})$ (Remark 1.3), and using Lemma,

$$\xi(\alpha, \beta) = \frac{\beta}{\lambda(1 - \mathbb{E}\,e^{-\beta\sigma}) - r\alpha + \beta} = \frac{\beta}{r\psi(\beta) - r\alpha}.$$

Method 2: ladder heights, ctd.

Next objective: compute

$$\eta(\alpha, \beta) := \mathbb{E}\left(e^{-\alpha H - \beta\tau}\, 1\{\tau < \infty\}\right).$$

Proposition

For any $\alpha \geqslant 0$ and $\beta > 0$,

$$\eta(\alpha, \beta) = 1 - \frac{\beta - \varphi(\alpha)}{r\psi(\beta) - r\alpha}.$$

Method 2: ladder heights, ctd.

Proof. Use decomposition

$$\frac{\beta}{\beta - \varphi(\alpha)} = \mathbb{E}\, e^{-\alpha Y(T_\beta)} = \eta_1(\alpha,\beta) + \eta_2(\alpha,\beta),$$

where

$$\eta_1(\alpha,\beta) := \mathbb{E}\Big(e^{-\alpha Y(T_\beta)}\, 1\{\tau > T_\beta, \tau < \infty\}\Big),$$
$$\eta_2(\alpha,\beta) := \mathbb{E}\Big(e^{-\alpha Y(T_\beta)}\, 1\{\tau \leqslant T_\beta, \tau < \infty\}\Big).$$

Recalling definition of $\xi(\alpha,\beta)$,

$$\begin{aligned}\eta_1(\alpha,\beta) &= \mathbb{E}\Big(e^{-\alpha Y(T_\beta)}\, 1\{\bar{Y}(T_\beta) = 0\}\Big)\\ &= \mathbb{E}\Big(e^{-\alpha Y(T_\beta)}\, 1\{Y(T_\beta) - \underline{Y}(T_\beta) = 0\}\Big) = \xi(\alpha,\beta),\end{aligned}$$

which is known from previous Proposition.

Method 2: ladder heights, ctd.

$$\eta_2(\alpha,\beta) = \int_{t=0}^{\infty} \beta e^{-\beta t} \int_{s=0}^{t} \int_{y=0}^{\infty} e^{-\alpha y}\, \mathbb{E}\big(e^{-\alpha(Y(t)-Y(\tau))} \,|\, H=y, \tau=s\big)$$
$$\mathbb{P}(H\in dy, \tau\in ds)\, dt$$
$$= \int_{t=0}^{\infty} \beta e^{-\beta t} \int_{s=0}^{t} \int_{y=0}^{\infty} e^{-\alpha y} e^{\varphi(\alpha)(t-s)}\, \mathbb{P}(H\in dy, \tau\in ds)\, dt.$$

Swap order of integrals:

$$\int_{s=0}^{\infty} \int_{y=0}^{\infty} \left(\int_{t=s}^{\infty} \beta e^{-\beta t} e^{\varphi(\alpha) t} dt \right) e^{-\alpha y}\, e^{-\varphi(\alpha) s}\, \mathbb{P}(H\in dy, \tau\in ds)$$
$$= \frac{\beta}{\beta-\varphi(\alpha)} \int_{s=0}^{\infty} \int_{y=0}^{\infty} e^{-\alpha y} e^{-\beta s}\, \mathbb{P}(H\in dy, \tau\in ds)$$
$$= \frac{\beta}{\beta-\varphi(\alpha)} \eta(\alpha,\beta).$$

Combining the above, stated follows after some algebra.

Method 2: ladder heights, ctd.

Using geometric-sum representation, we can now compute transform of running maximum $\bar{Y}(T_\beta)$. We thus recover time-dependent Pollaczek-Khinchine theorem:

$$\begin{aligned}
\varrho(\alpha,\beta) &= \sum_{k=0}^{\infty} \big(\eta(\alpha,\beta)\big)^k \big(1-\eta(0,\beta)\big) \\
&= \sum_{k=0}^{\infty} \left(1-\frac{\beta-\varphi(\alpha)}{r\psi(\beta)-r\alpha}\right)^k \frac{\beta}{r\psi(\beta)} \\
&= \frac{\alpha-\psi(\beta)}{\varphi(\alpha)-\beta}\,\frac{\beta}{\psi(\beta)}.
\end{aligned}$$

Method 3: Kella-Whitt martingale

Roadmap:

- Use queueing representation.
- Consider Kella-Whitt martingale involving the queueing process.
- By ‘optional sampling’ expression for $\varrho(\alpha, \beta)$ is found.

Method 3: Kella-Whitt martingale, ctd.

Define

$$M(t) := \varphi(\alpha)\int_0^t e^{-\alpha Q(s)}\,ds + 1 - e^{-\alpha Q(t)} + \alpha\underline{Y}(t).$$

Lemma

The process $M(t)$ is a martingale with respect to $\mathscr{F}(t)$, i.e., the natural filtration pertaining to $\{Y(s) : s \in [0,t]\}$.

Proof in e.g. Kyprianou book; informal support in Section 1.5.

Method 3: Kella-Whitt martingale, ctd.

Using 'optional sampling' with the stopping time T_β and recalling that $Q(0) = 0$, we have that $0 = \mathbb{E}\, M(0) = \mathbb{E}\, M(T_\beta)$.

Hence,

$$0 = \varphi(\alpha)\, \mathbb{E} \int_0^{T_\beta} e^{-\alpha Q(s)}\, ds + 1 - \mathbb{E}\, e^{-\alpha Q(T_\beta)} + \alpha\, \mathbb{E} \underline{Y}(T_\beta).$$

Method 3: Kella-Whitt martingale, ctd.

Swapping the order of integration,

$$
\begin{aligned}
\mathbb{E} \int_0^{T_\beta} e^{-\alpha Q(s)}\, ds &= \int_0^\infty \beta e^{-\beta t} \int_0^t \mathbb{E}\, e^{-\alpha Q(s)}\, ds\, dt \\
&= \int_0^\infty \left(\int_s^\infty \beta e^{-\beta t} dt \right) \mathbb{E}\, e^{-\alpha Q(s)}\, ds \\
&= \int_0^\infty e^{-\beta s} \mathbb{E}\, e^{-\alpha Q(s)}\, ds \\
&= \frac{1}{\beta}\, \mathbb{E}\, e^{-\alpha Q(T_\beta)}.
\end{aligned}
$$

Solving $\mathbb{E}\, e^{-\alpha Q(T_\beta)}$,

$$
\mathbb{E}\, e^{-\alpha Q(T_\beta)} = \frac{\beta}{\varphi(\alpha) - \beta} \left(-\alpha\, \mathbb{E} \underline{Y}(T_\beta) - 1 \right).
$$

Method 3: Kella-Whitt martingale, ctd.

Left: find $\mathbb{E}\underline{Y}(T_\beta)$. Note that (fixing $\beta > 0$) any root $\alpha > 0$ of denominator should be root of numerator as well.

Hence, using that $\alpha = \psi(\beta)$ is root of denominator,

$$-\psi(\beta)\,\mathbb{E}\underline{Y}(T_\beta) = 1,$$

so that $\mathbb{E}\underline{Y}(T_\beta) = -1/\psi(\beta)$.

From the above we conclude that, in agreement with time-dependent Pollaczek-Khinchine theorem,

$$\mathbb{E}\,e^{-\alpha Q(T_\beta)} = \frac{\alpha - \psi(\beta)}{\varphi(\alpha) - \beta}\,\frac{\beta}{\psi(\beta)}.$$

Method 4: Kolmogorov forward equations

Roadmap:

- Use queueing representation. Express $\mathbb{E}\, e^{-\alpha Q(t+\Delta t)}$ in terms of $\mathbb{E}\, e^{-\alpha Q(t)}$.
- Set up a differential equation, and transform it with respect to time.
- Solve the resulting identity, to obtain $\varrho(\alpha, \beta)$.

Method 4: Kolmogorov forward equations, ctd.

Define $F_t(y)$ as probability that $Q(t)$ does not exceed y, where $Q(0) = 0$, and let $f_t(y)$ denote corresponding density.

Elementary Δt-argument gives, up to $o(\Delta t)$-terms,

$$F_{t+\Delta t}(y) = F_t(y + r\,\Delta t)(1 - \lambda\,\Delta t) \\ + \lambda\,\Delta t \left(\int_{0+}^{y} f_t(z)\,\mathbb{P}(B \leqslant y - z)\,dz + F_t(0)\,\mathbb{P}(B \leqslant y) \right).$$

Subtracting $F_t(y + r\,\Delta t)$, dividing by Δt and letting $\Delta t \downarrow 0$:

$$\frac{\partial}{\partial t} F_t(y) = r f_t(y) - \lambda F_t(y) \\ + \lambda \left(\int_{0+}^{y} f_t(z)\,\mathbb{P}(B \leqslant y - z)\,dz + F_t(0)\,\mathbb{P}(B \leqslant y) \right).$$

Method 4: Kolmogorov forward equations, ctd.

Same can be done for LST of $Q(t)$. With

$$\kappa_t(\alpha) := \mathbb{E}\, e^{-\alpha Q(t)}, \quad \bar{\kappa}_t(\alpha) := \mathbb{E}\, e^{-\alpha Q(t)} 1\{Q(t) > 0\} = \kappa_t(\alpha) - q_t,$$

where $q_t := \mathbb{P}(Q(t) = 0) = F_t(0)$ and $1\{A\}$ indicator of event A,

$$\begin{aligned}
\bar{\kappa}_{t+\Delta t}(\alpha) + q_{t+\Delta t} &= \kappa_{t+\Delta t}(\alpha) \\
&= \bar{\kappa}_t(\alpha)\big(1 - \lambda\Delta t + \lambda\Delta t\, b(\alpha) + r\alpha\Delta t\big) + q_t\left(1 - \lambda\Delta t + \lambda\Delta t\, b(\alpha)\right) \\
&= \bar{\kappa}_t(\alpha)\big(1 + \varphi(\alpha)\Delta t\big) + \big(1 - \lambda\Delta t + \lambda\Delta t\, b(\alpha)\big) q_t.
\end{aligned}$$

Lemma

For any $\alpha, t > 0$,

$$\frac{\partial}{\partial t}\bar{\kappa}_t(\alpha) + \frac{\partial}{\partial t} q_t = \varphi(\alpha)\,\bar{\kappa}_t(\alpha) - q_t\,\lambda\left(1 - b(\alpha)\right).$$

Method 4: Kolmogorov forward equations, ctd.

Consider differential equation of Lemma, but now at exponentially distributed time T_β.

Standard identity

$$\int_0^\infty \beta e^{-\beta t} \left(\frac{\partial}{\partial t} f(t) \right) dt = -\beta f(0) + \beta \int_0^\infty \beta e^{-\beta t} f(t)\, dt.$$

Hence,

$$-\beta \bar{\kappa}_0(\alpha) + \beta \int_0^\infty \beta e^{-\beta t} \bar{\kappa}_t(\alpha)\, dt - \beta q_0 + \beta \int_0^\infty \beta e^{-\beta t} q_t\, dt$$
$$= \varphi(\alpha) \int_0^\infty \beta e^{-\beta t} \bar{\kappa}_t(\alpha)\, dt - \lambda \big(1 - b(\alpha)\big) \int_0^\infty \beta e^{-\beta t} q_t\, dt.$$

Method 4: Kolmogorov forward equations, ctd.

Due to $Q(0) = 0$, we have $\bar{\kappa}_0(\alpha) = 0$ and $q_0 = 1$. Rearranging, and using definition of $\varphi(\alpha)$,

$$\big(\beta - \varphi(\alpha)\big)\bar{\kappa}_{T_\beta}(\alpha) + \big(\beta - \varphi(\alpha) + r\alpha\big)q_{T_\beta} = \beta.$$

Observe that q_{T_β} can be identified by inserting $\alpha = \psi(\beta)$:

$$q_{T_\beta} = \frac{\beta}{\beta + \lambda(1 - b(\psi(\beta)))} = \frac{\beta}{r\psi(\beta)}.$$

Time-dependent Pollaczek-Khinchine theorem is recovered:

$$\begin{aligned}\mathbb{E}\, e^{-\alpha Q(T_\beta)} &= \bar{\kappa}_{T_\beta}(\alpha) + q_{T_\beta} = \frac{\beta}{\beta - \varphi(\alpha)} - \frac{\beta - \varphi(\alpha) + r\alpha}{\beta - \varphi(\alpha)} q_{T_\beta} + q_{T_\beta} \\ &= \frac{\beta - r\alpha\, q_{T_\beta}}{\beta - \varphi(\alpha)} = \frac{\alpha - \psi(\beta)}{\varphi(\alpha) - \beta}\frac{\beta}{\psi(\beta)}.\end{aligned}$$

Method 4: Kolmogorov forward equations, ctd.

Interesting connection with concept of *rate conservation*.
We get back to this in Chapter 5.

Yields elegant way to show that stationary workload $Q(\infty)$ is distributed as sum of geometric number (with success probability c) of i.i.d. copies of $\bar{B}$.

Chapter 1: concluding remarks

In Exercise 1.2 you will substantially generalize result on $\pi(\alpha, \beta)$.

Instead of looking at ruin probabilities, we consider object, with $\boldsymbol{\gamma} := (\gamma_1, \gamma_2, \gamma_3)$,

$$p(u, t, \boldsymbol{\gamma}) := \mathbb{E}\left(e^{-\gamma_1 \tau(u) - \gamma_2 X_u(\tau(u)-) - \gamma_3 X_u(\tau(u))} 1\{\tau(u) \leqslant t\}\right).$$

This includes time of ruin $\tau(u)$, value of reserve process *immediately before* ruin $X_u(\tau(u)-)$, and value of reserve process *at* ruin $X_u(\tau(u))$. Here $X_u(\tau(u)-) > 0$ can be seen as *undershoot*, and $-X_u(\tau(u)) \geqslant 0$ as *overshoot*.

Chapter 1: concluding remarks

In Exercise 1.5 you will establish second substantial generalization.

Brownian component is included into net cumulative claim process $Y(t)$. Remarkably, results for CL model (i.e., *without* Brownian component) can still be used when describing transform of $\bar{Y}(\infty)$.

CHAPTER II: ASYMPTOTICS

Asymptotics: main ideas

- Previous chapter: we derived *transform* of ruin probability.
- Interestingly, when settling for stationary asymptotics (i.e., $\mathbb{P}(\bar{Y}(\infty) > u)$ for u large) explicit results *can* be found.
- Need to distinguish between light-tailed and heavy-tailed claim-size distributions.
- Transient asymptotics (i.e., $\mathbb{P}(Y(t) > u)$ for u large) harder to analyse.

Asymptotics: main ideas, ctd.

We throughout assume $\mathbb{E}\, Y(1) < 0$, so that $Y(t)$ does not drift to ∞ as $t \to \infty$. Hence $p(u) \to 0$ as $u \to \infty$.

Equivalently: impose *net-profit condition* $\lambda\, \mathbb{E} B < r$.

Distinguish between claim-size distribution having a light or heavy tail:

- Light-tailed case: $p(u)$ decays exponentially, with (for large u) net cumulative claim process moving 'roughly gradually' towards level u.
- Heavy-tailed ('subexponential') case: exceeding level u is (for large u) with overwhelming probability due to a single large claim.

Asymptotics: main ideas, ctd.

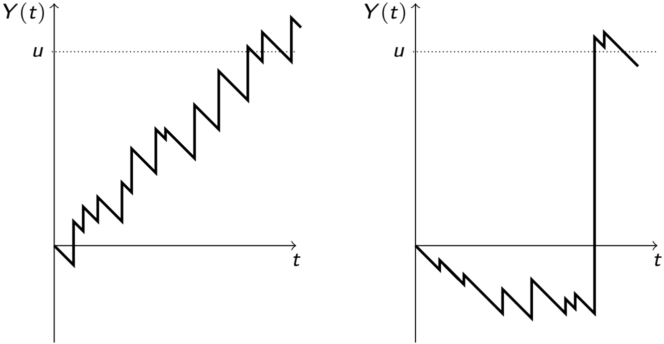

Figure: Typical trajectory of net cumulative claim process $Y(t)$ exceeding high level u in light-tailed case (left panel), and in heavy-tailed case (right panel).

Light-tailed case

Assume: there is strictly positive solution $\theta^\star$ to the equation $\varphi(-\theta^\star) = 0$; recall

$$\varphi(\alpha) := \log \mathbb{E} e^{-\alpha Y(1)} = r\alpha - \lambda(1 - b(\alpha)).$$

This requires $\mathbb{E} e^{-\alpha Y(1)} < \infty$ for some $\alpha < 0$, and therefore all moments of $Y(1)$ are finite (and hence all moments of $Y(t)$ for any $t \geqslant 0$).

Explains why we refer to this setting as *light-tailed case*. It implicitly means that claim size B is light-tailed as well; write $B \in \mathscr{L}$.

Primary objective: identify *exact asymptotics* of $p(u)$ for $B \in \mathscr{L}$: we find explicit function $\hat{p}(u)$ such that $p(u)/\hat{p}(u) \to 1$ as $u \to \infty$.

Light-tailed case, ctd.

Change-of-measure: work with net cumulative claim process with Laplace exponent $\varphi_{\mathbb{Q}}(\alpha) := \varphi(\alpha - \theta^\star)$ rather than $\varphi(\alpha)$.
$\mathbb{Q}$: probability measure that goes with this alternative Laplace exponent.

Next goal: check that $\varphi_{\mathbb{Q}}(\alpha)$ is indeed Laplace exponent of compound Poisson process with drift.

Light-tailed case, ctd.

As $\theta^\star$ solves the equation $-r\theta^\star - \lambda(1 - b(-\theta^\star)) = 0$, we can write

$$\begin{aligned}\varphi(\alpha - \theta^\star) &= r(\alpha - \theta^\star) - \lambda(1 - b(\alpha - \theta^\star)) \\ &= r\alpha - \lambda b(-\theta^\star)\left(1 - \frac{b(\alpha - \theta^\star)}{b(-\theta^\star)}\right).\end{aligned}$$

Note: $e^{\theta^\star x}\,\mathbb{P}(B \in dx)/b(-\theta^\star)$ is a density of random variable with LST $b(\alpha - \theta^\star)/b(-\theta^\star)$.
We say: this density is an *exponentially twisted* version of original density.

Conclude: $\varphi_{\mathbb{Q}}(\alpha) = \varphi(\alpha - \theta^\star)$ is Laplace exponent of compound Poisson process where

- claim arrival rate is $\lambda_{\mathbb{Q}} := \lambda b(-\theta^\star)$,
- claims have LST $b_{\mathbb{Q}}(\alpha) := b(\alpha - \theta^\star)/b(-\theta^\star)$,
- negative drift remains r.

Light-tailed case, ctd.

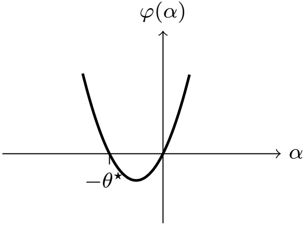

Figure: Functions $\varphi(\alpha)$ (left panel) and $\varphi_{\mathbb{Q}}(\alpha)$ (right panel). Observe: $\varphi'(0) > 0$ but $\varphi'_{\mathbb{Q}}(0) < 0$.

Light-tailed case, ctd.

Recall: $Y(t)$ drifts to $-\infty$ under $\mathbb{P}$, due to $\mathbb{E}\,Y(1) < 0$. And under $\mathbb{Q}$?

First note:

$$\mathbb{E}_{\mathbb{Q}} B = -b'_{\mathbb{Q}}(0) = -\frac{b'(-\theta^\star)}{b(-\theta^\star)},$$

with $\mathbb{E}_{\mathbb{Q}}(\cdot)$ denoting expectation under $\mathbb{Q}$.

Hence,

$$\mathbb{E}_{\mathbb{Q}} Y(1) = \lambda_{\mathbb{Q}} \left(-\frac{b'(-\theta^\star)}{b(-\theta^\star)}\right) - r = -\lambda\, b'(-\theta^\star) - r = -\varphi'(-\theta^\star) = -\varphi'_{\mathbb{Q}}(0),$$

which is positive due to convexity of $\varphi(\cdot)$ and definition of $\theta^\star$.
Conclude: under $\mathbb{Q}$ process $Y(\cdot)$ drifts to ∞.

Light-tailed case, ctd.

Main idea behind finding exact asymptotics of $p(u)$ using $\mathbb{Q}$:

- Denote by $\tau(u)$ first time that $Y(\cdot)$ reaches u. Hence, $p(u) = \mathbb{P}(\tau(u) < \infty)$.
- Perform experiment of verifying whether or not $\tau(u) < \infty$ applies under $\mathbb{Q}$ rather than under $\mathbb{P}$.
- Under $\mathbb{Q}$ event $\{\tau(u) < \infty\}$ has probability 1, due to $\mathbb{E}_{\mathbb{Q}} Y(1) > 0$, but apply 'compensation' to correct for difference between $\mathbb{P}$ and $\mathbb{Q}$.
- Use results from Section 1.4 to derive exact asymptotics.

Light-tailed case, ctd.

N: index of claim at which, in our experiment, u is reached.

Hence, at that point interarrival times (say) $\boldsymbol{E} \equiv (E_1, \ldots, E_N)$ and claim sizes $\boldsymbol{B} \equiv (B_1, \ldots, B_N)$ have been sampled (under $\mathbb{Q}$).

With $L \equiv L(\boldsymbol{E}, \boldsymbol{B})$ denoting likelihood ratio of $(\boldsymbol{E}, \boldsymbol{B})$ (under $\mathbb{P}$, relative to $\mathbb{Q}$), we have identity

$$p(u) = \mathbb{P}(\tau(u) < \infty) = \mathbb{E}1\{\tau(u) < \infty\} = \mathbb{E}_{\mathbb{Q}}\big(1\{\tau(u) < \infty\}\, L(\boldsymbol{E}, \boldsymbol{B})\big).$$

Here $L(\boldsymbol{E}, \boldsymbol{B})$ is *Radon-Nikodym derivative*, often denoted by

$$L = \frac{d\mathbb{P}}{d\mathbb{Q}} \equiv \frac{d\mathbb{P}}{d\mathbb{Q}}(\boldsymbol{E}, \boldsymbol{B}).$$

Light-tailed case, ctd.

L: ratio of the densities of all sampled quantities, where in numerator these correspond to $\mathbb{P}$ and in denominator to $\mathbb{Q}$.

With $f_{\mathbb{P}}(\cdot)$ and $f_{\mathbb{Q}}(\cdot)$ the densities of B under $\mathbb{P}$ and $\mathbb{Q}$,

$$L(\boldsymbol{E}, \boldsymbol{B}) = \frac{\lambda\, e^{-\lambda E_1}\, f_{\mathbb{P}}(B_1) \cdots \lambda\, e^{-\lambda E_N}\, f_{\mathbb{P}}(B_N)}{\lambda_{\mathbb{Q}}\, e^{-\lambda_{\mathbb{Q}} E_1}\, f_{\mathbb{Q}}(B_1) \cdots \lambda_{\mathbb{Q}}\, e^{-\lambda_{\mathbb{Q}} E_N}\, f_{\mathbb{Q}}(B_N)}.$$

Applying

$$\frac{\lambda}{\lambda_{\mathbb{Q}}} = \frac{1}{b(-\theta^\star)}, \qquad \frac{f_{\mathbb{P}}(x)}{f_{\mathbb{Q}}(x)} = e^{-\theta^\star x} b(-\theta^\star),$$

expression for $L(\boldsymbol{E}, \boldsymbol{B})$ can be rewritten.

Light-tailed case, ctd.

$$L(\boldsymbol{E}, \boldsymbol{B}) = \exp\left((\lambda_{\mathbb{Q}} - \lambda) \sum_{n=1}^{N} E_n - \theta^\star \sum_{n=1}^{N} B_n\right)$$
$$= \exp\left(-\lambda(1 - b(-\theta^\star)) \sum_{n=1}^{N} E_n - \theta^\star \sum_{n=1}^{N} B_n\right)$$
$$= \exp\left(r\theta^\star \sum_{n=1}^{N} E_n - \theta^\star \sum_{n=1}^{N} B_n\right) = e^{-\theta^\star Y(\tau(u))}.$$

Recall $p(u) = \mathbb{E}_{\mathbb{Q}}(1\{\tau(u) < \infty\}\, L(\boldsymbol{E}, \boldsymbol{B}))$ and $\mathbb{Q}(\tau(u) < \infty) = 1$.

Proposition

Assume $B \in \mathscr{L}$. For any $u > 0$,

$$p(u) = \mathbb{E}_{\mathbb{Q}} e^{-\theta^\star Y(\tau(u))}.$$

Light-tailed case, ctd.

Previous Proposition holds for any $u > 0$; no 'asymptotics'.

Realizing that (by definition) $Y(\tau(u)) \geqslant u$, following upper bound follows.

Proposition (Lundberg inequality)

Assume $B \in \mathscr{L}$. For any $u > 0$,

$$p(u) \leqslant e^{-\theta^\star u}.$$

Light-tailed case, ctd.

Observe that we can write $Y(\tau(u)) = u + R(u)$, with $R(u) \geqslant 0$ *overshoot* over level u. Idea: prove that $\mathbb{E}_{\mathbb{Q}} e^{-\theta^\star R(u)} \to \gamma$, as $u \to \infty$, which then implies that

$$\lim_{u\to\infty} p(u)\, e^{\theta^\star u} \to \gamma.$$

$(H_n)_n$: ladder height process corresponding to net cumulative claim process $Y(t)$ (see Section 1.4).

Individual ladder heights are i.i.d., so that $(H_n)_n$ is *renewal process* (which is, under $\mathbb{Q}$, non-defective); let H denote generic ladder height.

Crucial observation: $R(u)$ is overshoot of $(H_n)_n$ over u.

Light-tailed case, ctd.

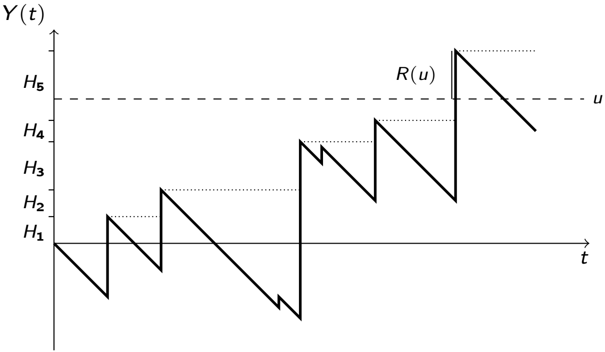

Figure: Net cumulative claim process $Y(t)$, ladder height process $(H_n)_n$, and overshoot $R(u)$ over level u (dashed line); corresponding running maximum process $\bar{Y}(t)$ is depicted by dotted lines.

Light-tailed case, ctd.

Renewal theory: as $u \to \infty$, overshoot converges (in distribution) to limiting variable $\bar{H}$ with distribution function

$$\mathbb{Q}(\bar{H} \leqslant x) = \int_0^x \frac{\mathbb{Q}(H \geqslant y)}{\mathbb{E}_{\mathbb{Q}} H} dy.$$

Conclude

$$\gamma = \mathbb{E}_{\mathbb{Q}} e^{-\theta^\star \bar{H}}.$$

Use theory developed in Section 1.4 to evaluate γ.

Light-tailed case, ctd.

First determine $\mathbb{E}_{\mathbb{Q}}\, e^{-\alpha H}$. Proposition 1.4:

$$\mathbb{E}_{\mathbb{Q}}\, e^{-\alpha H} = 1 - \frac{0 - \varphi_{\mathbb{Q}}(\alpha)}{r\theta^\star - r\alpha} = \frac{\lambda}{r}\frac{1 - b(\alpha - \theta^\star)}{\alpha - \theta^\star};$$

use that, in self-evident notation, $\psi_{\mathbb{Q}}(0) = \theta^\star$. Then,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}} H &= -\lim_{\alpha\downarrow 0}\frac{d}{d\alpha}\mathbb{E}_{\mathbb{Q}}\, e^{-\alpha H} = \frac{\lambda}{r}\lim_{\alpha\downarrow 0}\frac{1 - b(\alpha - \theta^\star) + (\alpha - \theta^\star) b'(\alpha - \theta^\star)}{(\alpha - \theta^\star)^2}\\ &= \frac{\lambda}{r}\frac{1 - b(-\theta^\star) - \theta^\star\, b'(-\theta^\star)}{(\theta^\star)^2} = \frac{1}{r\theta^\star}\mathbb{E}_{\mathbb{Q}} Y(1)\end{aligned}$$

(last equality: use that $\theta^\star$ solves $\varphi(-\theta^\star) = 0$ and definition of $\mathbb{E}_{\mathbb{Q}} Y(1)$).

Light-tailed case, ctd.

By definition of $\bar{H}$,

$$\mathbb{E}_{\mathbb{Q}}\, e^{-\alpha \bar{H}} = \frac{1}{\alpha} \frac{1}{\mathbb{E}_{\mathbb{Q}} H} \left(1 - \mathbb{E}_{\mathbb{Q}}\, e^{-\alpha H}\right),$$

so that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\, e^{-\theta^\star \bar{H}} &= \lim_{\alpha \downarrow \theta^\star} \frac{1}{\alpha} \frac{1}{\mathbb{E}_{\mathbb{Q}} H} \left(1 - \mathbb{E}_{\mathbb{Q}}\, e^{-\alpha H}\right) \\
&= \frac{1}{\theta^\star} \frac{1}{\mathbb{E}_{\mathbb{Q}} H} \left(1 + \frac{\lambda}{r} b'(0)\right) = -\frac{1}{r\theta^\star} \frac{1}{\mathbb{E}_{\mathbb{Q}} H} \mathbb{E}\, Y(1).
\end{aligned}$$

Conclude: $\gamma = -\mathbb{E}\, Y(1)/\mathbb{E}_{\mathbb{Q}} Y(1) \in (0, \infty)$.

Light-tailed case, ctd.

Theorem (Cramér-Lundberg approximation)

Assume $B \in \mathscr{L}$. As $u \to \infty$,

$$p(u)\, e^{\theta^\star u} \to \gamma := -\frac{\mathbb{E}\,Y(1)}{\mathbb{E}_{\mathbb{Q}}\,Y(1)}.$$

In practice we use, for u large,

$$p(u) \approx \hat{p}(u) := \gamma e^{-\theta^\star u}.$$

Exercise 2.3: you will extend this result to case where Brownian motion has been added to net cumulative claim process $Y(t)$.

Light-tailed case, ctd.

In case one settles for just exponential decay rate $\theta^\star$, elementary derivation can be given, using *large deviation theory*.

- Let $Y_1, Y_2, \ldots$ be i.i.d. random variables distributed as $Y(1)$. *Cramér's theorem*: for $a > \mathbb{E}Y(1)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\sum_{i=1}^{n} Y_i \geqslant na\right) = -I(a),$$

  where $I(a) := \sup_{\theta>0}(\theta a - \varphi(-\theta))$ is *Legendre transform* of the cumulant generating function $\varphi(-\theta)$. $I(a)$ is non-negative and convex, and attains its minimal value 0 in $a = \mathbb{E}Y(1) = -\varphi'(0)$.
- *Chernoff bound*: uniformly in n,

$$\mathbb{P}\left(\sum_{i=1}^{n} Y_i \geqslant na\right) \leqslant e^{-nI(a)}.$$

Light-tailed case, ctd.

Lower bound: for any $T > 0$, $p(u) = \mathbb{P}(\bar{Y}(\infty) \geqslant u) \geqslant \mathbb{P}(Y(Tu) \geqslant u)$. Hence, for all $T, u > 0$,

$$\frac{1}{u} \log p(u) \geqslant \frac{T}{Tu} \log \mathbb{P}\left(\frac{Y(Tu)}{Tu} \geqslant \frac{1}{T}\right).$$

Applying Cramér's theorem:

$$\liminf_{u\to\infty} \frac{1}{u} \log p(u) \geqslant -T\, I(1/T).$$

As this lower bound applies to any $T > 0$, we can select *sharpest* lower bound. Denoting $I^\star := T^\star I(1/T^\star)$ with $T^\star := \arg\inf_{T>0} T\, I(1/T)$,

$$\liminf_{u\to\infty} \frac{1}{u} \log p(u) \geqslant -I^\star.$$

Later we'll show $I^\star = \theta^\star$.

Light-tailed case, ctd.

Upper bound: first observe that (Why?)

$$p(u) \leqslant \mathbb{P}\left(\exists n \in \mathbb{N} : \sum_{i=1}^{n} Y_i \geqslant u - r\right) \leqslant \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} Y_i \geqslant u - r\right).$$

For given $\varepsilon > 0$, split into two sums:

$$\sum_{n=1}^{T^\star(1+\varepsilon)u} \mathbb{P}\left(\sum_{i=1}^{n} Y_i \geqslant u - r\right) + \sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} Y_i \geqslant u - r\right);$$

For $u > r$, second sum is dominated by (Chernoff bound!)

$$\sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} Y_i \geqslant 0\right) \leqslant \sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} e^{-nI(0)} = \frac{e^{-(T^\star(1+\varepsilon)u+1)I(0)}}{1 - e^{-I(0)}}.$$

Light-tailed case, ctd.

First sum is majorized by (again Chernoff bound!)

$$T^\star(1+\varepsilon)u \max_{n=1,\ldots,T^\star(1+\varepsilon)u} \mathbb{P}\left(\sum_{i=1}^{n} Y_i \geqslant u-r\right)$$
$$\leqslant T^\star(1+\varepsilon)u \max_{n=1,\ldots,T^\star(1+\varepsilon)u} \exp\left(-nI\left(\frac{u-r}{n}\right)\right).$$

By definition of $T^\star$, for any $\delta > 0$ and u large enough

$$\exp\left(-nI\left(\frac{u-r}{n}\right)\right) = \exp\left(-(u-r)\frac{n}{u-r}I\left(\frac{u-r}{n}\right)\right) \leqslant e^{-(u-r)(I^\star-\delta)}$$

for all $n \in \{1,\ldots,T^\star(1+\varepsilon)u\}$.

Pick ε large enough that $T^\star(1+\varepsilon)I(0) > I^\star - \delta$, so that decay rate of first sum dominates.

Light-tailed case, ctd.

Conclude:

$$\limsup_{u\to\infty} \frac{1}{u} \log p(u) \leqslant \limsup_{u\to\infty} \frac{1}{u} \log \left(T^\star (1+\varepsilon) u \, e^{-(u-r)(I^\star-\delta)} \right)$$
$$= -I^\star + \delta.$$

Let $\delta \downarrow 0$.

Together with the lower bound: *logarithmic asymptotics* of $p(u)$ are given by

$$\lim_{u\to\infty} \frac{1}{u} \log p(u) = -I^\star.$$

Left to prove: $I^\star = \theta^\star$.

Light-tailed case, ctd.

Let $\theta(a)$ be optimizing argument in definition of $I(a)$, i.e., $\theta(a)$ solves $a = -\varphi'(-\theta)$.

Define $\Delta := 1/T$ so that $I^\star = \inf_{\Delta>0} I(\Delta)/\Delta$. To find optimizing Δ, compute derivative of $I(\Delta)/\Delta$ and equate it to 0. First order condition: $\Delta I'(\Delta) - I(\Delta) = 0$, and hence

$$\Delta\big(\theta'(\Delta)\Delta + \theta(\Delta) + \varphi'(-\theta(\Delta))\theta'(\Delta)\big) - I(\Delta) = 0.$$

But, as we know that $\Delta + \varphi'(-\theta(\Delta)) = 0$, this condition reduces to $\Delta\theta(\Delta) = I(\Delta)$, i.e., $\varphi(-\theta(\Delta^\star)) = 0$ for optimizing $\Delta^\star$.

Hence, $\theta(\Delta^\star) = \theta^\star$. Conclude

$$I^\star = \frac{I(\Delta^\star)}{\Delta^\star} = \frac{\Delta^\star\,\theta^\star}{\Delta^\star} = \theta^\star.$$

Light-tailed case, ctd.

Minimization $\inf_{\Delta>0} I(\Delta)/\Delta$ has following appealing interpretation.

Δ: slope at which $Y(t)$ moves from level 0 to level u, which 'costs' $I(\Delta)$ per unit of time. Time needed to reach u is proportional to $1/\Delta$.

When optimizing cost $I(\Delta)/\Delta$ over Δ, we obtain 'cheapest' slope. Trade-off: low Δ leads to low cost per unit of time but then unusual behavior has to persist for long time, and vice versa for high Δ.

Timescale $T^\star := 1/\Delta^\star$ has similar interpretation: $T^\star u$ is typical time to reach u. (In proof: first sum, containing timescales around $T^\star u$, dominates second sum.)

Subexponential case

Result from Chapter 1:

$$p(u) = \mathbb{P}\left(\sum_{i=1}^{G} \bar{B}_i \geqslant u\right) = \mathbb{P}\left(\bar{B}^{\star G} \geqslant u\right),$$

where $\bar{B}$ is 'residual' of B, and G is geometric with success probability

$$c := 1 - \lambda\,\mathbb{E}B/r.$$

The density of $\bar{B}$ is given by

$$f_{\bar{B}}(t) := \frac{\mathbb{P}(B \geqslant t)}{\mathbb{E}B}.$$

Subexponential case, ctd.

In previous section: claim-size distribution was light-tailed (so that all moments exist).
Now: what happens if this condition is violated?

We assume that $\bar{B}$ is such that, as $u \to \infty$,

$$\frac{\mathbb{P}(\bar{B}^{\star 2} \geqslant u)}{\mathbb{P}(\bar{B} \geqslant u)} \to 2.$$

(If the sum of two i.i.d. copies of $\bar{B}$ is large, this is due to one of them being large, rather than both of them significantly contributing.)

Write: $\bar{B} \in \mathscr{S}$ with $\mathscr{S}$ set of *subexponential distributions*.

In general neither $\bar{B} \in \mathscr{S}$ implies $B \in \mathscr{S}$, nor $B \in \mathscr{S}$ implies $\bar{B} \in \mathscr{S}$. But, for broad set of relevant distributions, $B \in \mathscr{S}$ and $\bar{B} \in \mathscr{S}$ are equivalent.

Subexponential case, ctd.

Theorem

Assume $\bar{B} \in \mathscr{S}$. As $u \to \infty$,

$$\frac{p(u)}{\mathbb{P}(\bar{B} \geqslant u)} \to \frac{1-c}{c}.$$

First: some auxiliary results, covering useful properties of subexponential distributions.

Subexponential case, ctd.

Lemma

(i) If $Y \in \mathscr{S}$, then, as $u \to \infty$

$$\frac{\mathbb{P}(Y^{\star i} \geqslant u)}{\mathbb{P}(Y \geqslant u)} \to i.$$

(ii) If $Y \in \mathscr{S}$, then for all $\varepsilon > 0$ there exists a constant K_ε such that, for all i and u,

$$\mathbb{P}(Y^{\star i} \geqslant u) \leqslant K_\varepsilon (1+\varepsilon)^i \, \mathbb{P}(Y \geqslant u).$$

(iii) Let $Y_1, Y_2, \ldots$ be i.i.d., distributed as generic random variable Y. Let $I \in \mathbb{N}_0$ be independent of $Y_1, Y_2, \ldots$ with $\mathbb{E}\, z^I < \infty$ for some $z > 1$. Then, as $u \to \infty$,

$$\frac{\mathbb{P}(Y^{\star I} \geqslant u)}{\mathbb{P}(Y \geqslant u)} \to \mathbb{E} I.$$

Subexponential case, ctd.

Proof. Part (i) follows inductively from definition.

Part (ii): see proof in book Asmussen & Albrecher.

Part (iii) is 'stochastic version' of part (i). Proof relies on bounded convergence; see proof Lemma 2.2.

Proof of Theorem. Combine geometric sum representation with part (iii) of Lemma. In addition, observe that

$$\mathbb{E}G = \sum_{i=0}^{\infty} i\,(1-c)^i c = \frac{1-c}{c}$$

and $\mathbb{E}\,z^G < \infty$ if $z \in (1, 1/(1-c))$. Conclude that

$$\frac{p(u)}{\mathbb{P}(\bar{B} \geqslant u)} \to \frac{1-c}{c}.$$

Subexponential case, ctd.

Examples of subexponential distributions: Pareto, lognormal, and Weibull. Then $\mathbb{P}(B \geqslant u)$ is given by, respectively,

$$\frac{A^\eta}{(A+u)^\eta}, \quad 1-\Phi\left(\frac{\log u-\mu}{\sigma}\right), \quad e^{-\mu u^\eta},$$

with $\Phi(\cdot)$ distribution function of standard normal random variable.

Assumptions imposed on parameters:

- In Pareto case: $A > 0$ and $\eta > 1$ (to ensure that $\mathbb{E}B < \infty$).
- In lognormal case: $\mu \in \mathbb{R}$ and $\sigma > 0$.
- In Weibull case: $\mu > 0$ and $\eta \in (0,1)$.

Book: argumentation that residuals of these distributions are subexponential as well.

Subexponential case, ctd.

Principle of single big claim: in subexponential regime, event $\{\bar{Y}(\infty) \geqslant u\}$, for u large, is essentially due to *single big claim*.

Informal backing:
Suppose u is to be exceeded at time t. Then $Y(t)$ is roughly at $-rct = -(r - \lambda\,\mathbb{E}B)t$, so that big claim arriving at time t should have size at least $u + rct$. Leads to approximation, for $\Delta \downarrow 0$,

$$p(u) \approx \sum_{k=0}^{\infty} \lambda\Delta\,\mathbb{P}(B > u + rc\,k\Delta) \to \lambda \int_0^{\infty} \mathbb{P}(B > u + rcs)\,ds.$$

This confirms Theorem: performing change of variable $v := u + rcs$,

$$p(u) \approx \frac{\lambda}{rc}\int_u^{\infty} \mathbb{P}(B \geqslant v)\,dv = \frac{\lambda\,\mathbb{E}B}{rc}\,\mathbb{P}(\bar{B} \geqslant u) = \frac{1-c}{c}\,\mathbb{P}(\bar{B} \geqslant u).$$

Subexponential case, ctd.

Exercise 2.5: other technique to find tail of $p(u)$ for $\bar{B}$ in subclass of $\mathscr{S}$, based on *Tauberian theorems* (one-to-one relation between shape of LST near origin and tail behavior, if tail of rv is effectively power-law).

Concretely, for $\delta \in (1,2)$, the following equivalence holds:

$$\lim_{\alpha \downarrow 0} \frac{\mathbb{E}\, e^{-\alpha Z} - 1 + \alpha\, \mathbb{E} Z}{\alpha^\delta} = \eta \iff \lim_{u \to \infty} \mathbb{P}(Z \geqslant u)\, u^\delta = -\frac{\eta}{\Gamma(1-\delta)};$$

here $\eta > 0$ and $\Gamma(1-\delta) < 0$. Likewise, for $\delta \in (0,1)$,

$$\lim_{\alpha \downarrow 0} \frac{\mathbb{E}\, e^{-\alpha Z} - 1}{\alpha^\delta} = -\eta \iff \lim_{u \to \infty} \mathbb{P}(Z \geqslant u)\, u^\delta = \frac{\eta}{\Gamma(1-\delta)};$$

here $\eta > 0$ and $\Gamma(1-\delta) > 0$. More general form involves *regularly varying functions*.

Subexponential case, ctd.

What about finite-horizon ruin probability? Focus on $p(u, tu)$ as $u \to \infty$, for given t.

- Light-tailed case: by large-deviations argumentation,

$$\lim_{u\to\infty} \frac{1}{u} \log p(u, tu) = -\inf_{T\in(0,t]} T\, I\left(\frac{1}{T}\right).$$

  (Provide intuitive backing.)

- Subexponential case: by principle of single big claim:

$$\begin{aligned} p(u, tu) &\approx \frac{\lambda}{rc} \int_u^{u(1+rct)} \mathbb{P}(B \geqslant v)\, dv \\ &= \frac{1-c}{c}\Big(\mathbb{P}(\bar{B} \geqslant u) - \mathbb{P}(\bar{B} \geqslant u(1+rct))\Big). \end{aligned}$$

Heavy traffic

Focus on behavior of $\bar{Y}(\infty)$ as $c = 1 - \lambda\,\mathbb{E}B/r \downarrow 0$ (in queueing literature *heavy-traffic regime*). In this regime *safety loading* $r/(\lambda\,\mathbb{E}B) - 1$ is positive but small.

Starting point: Pollaczek-Khinchine formula of Corollary 1.1, i.e.,

$$\mathbb{E}\, e^{-\alpha \bar{Y}(\infty)} = \frac{rc\alpha}{r\alpha - \lambda(1 - b(\alpha))}.$$

Distinguish between light-tailed setting ($\mathbb{V}\mathrm{ar}\, B < \infty$) and heavy-tailed setting ($\mathbb{V}\mathrm{ar}\, B = \infty$). Write $\bar{Y}_c(\infty)$ rather than $\bar{Y}(\infty)$.

Heavy traffic, ctd.

First case $\mathbb{V}\mathrm{ar}\, B < \infty$. Clearly $\bar{Y}_c(\infty)$ explodes as $c \downarrow 0$, but $c\,\bar{Y}_c(\infty)$ converges to non-degenerate limiting random variable:

$$\begin{aligned}
\mathbb{E}\, e^{-c\alpha \bar{Y}_c(\infty)} &= \frac{rc^2\alpha}{rc\alpha - \lambda(1 - b(c\alpha))} \\
&= \frac{rc^2\alpha}{rc\alpha - \lambda\Big(\mathbb{E}B\, c\alpha - \frac{1}{2}\mathbb{E}[B^2]\, c^2\alpha^2 + O(c^3)\Big)} \\
&= \frac{rc^2\alpha}{rc\alpha - r(1-c)c\alpha + \frac{1}{2}\lambda\mathbb{E}[B^2]\, c^2\alpha^2 + O(c^3)} \\
&\to \frac{r}{r + \frac{1}{2}\lambda\,\mathbb{E}[B^2]\,\alpha},
\end{aligned}$$

as $c \downarrow 0$. *Lévy's convergence theorem*: conclude that $c\,\bar{Y}_c(\infty)$ converges to exponentially distributed random variable with mean

$$\frac{\lambda\mathbb{E}[B^2]}{2r} \xrightarrow{c\downarrow 0} \frac{\mathbb{E}[B^2]}{2\,\mathbb{E}B}.$$

Heavy traffic, ctd.

Case $\mathbb{V}\mathrm{ar}\, B = \infty$ (or, equivalently, $\mathbb{E}[B^2] = \infty$) should be done differently. Consider special case that, for some $\delta \in (1,2)$ and $A > 0$,

$$\mathbb{P}(B \geqslant u) \sim -\frac{A}{\Gamma(1-\delta)}\, u^{-\delta}$$

as $u \to \infty$. Tauber: as $\alpha \downarrow 0$,

$$b(\alpha) - 1 + \alpha\, \mathbb{E}B \sim A\, \alpha^{\delta}.$$

Now, with $\zeta := 1/(\delta - 1)$, $c^{\zeta}\, \bar{Y}_c(\infty)$ converges to a non-degenerate random variable:

$$\mathbb{E}\, e^{-c^{\zeta}\alpha \bar{Y}_c(\infty)} = \frac{rc^{1+\zeta}\alpha}{rc^{\zeta}\alpha - \lambda(1 - b(c^{\zeta}\alpha))} \to \frac{r}{r + \lambda A\, \alpha^{\delta-1}},$$

as $c \downarrow 0$. Recognize LST of *Mittag-Leffler distribution*.

CHAPTER III: REGIME SWITCHING

Regime switching: main ideas

- Previous chapters: a given net cumulative claim process $Y(t)$ was considered.
- Now: when exogenous finite-state Markov chain is in state i, net cumulative claim process behaves as $Y_i(t)$.
- Extension of time-dependent Pollaczek-Khinchine theorem.
- By-product: ruin probability with phase-type (rather than exponential) killing.

Regime switching: net cumulative claim process

This chapter: regime-switching (or Markov modulated) version of the standard Cramér-Lundberg model.

Modulating process, in our case a continuous-time Markov chain $J(t)$ on $\{1, \dots, d\}$: *regime process* or *background process*.

There are d net cumulative claim processes $Y_i(t)$, all of them corresponding to a compound Poisson process with drift.

Net cumulative claim process $Y(t)$ evolves as process $Y_i(t)$ when $J(t) = i$.

Regime switching: net cumulative claim process, ctd.

Description of *regime process*:

- $J(t)$: Markov process with transition rate matrix Q.
- $(U_n)_n$: sequence of its jump epochs.
- We do *not* assume modulating process is irreducible.
- In addition,
$$q_i := -q_{ii} = \sum_{j \neq i} q_{ij} > 0$$
for all non-absorbing states i, whereas $q_i := -q_{ii} = 0$ for absorbing states i.

Regime switching: net cumulative claim process, ctd.

Description of *net cumulative claim process*:

- $Y_1(t), \ldots, Y_d(t)$: independent compound Poisson processes with drift, evolving independently of $J(t)$.
- Recall: $(U_n)_n$ is sequence of its jump epochs.
- Laplace exponent of $Y_i(t)$ is

$$\varphi_i(\alpha) := r_i\alpha - \lambda_i\left(1 - \mathbb{E}\, e^{-\alpha B^{(i)}}\right) = r_i\alpha - \lambda_i(1 - b_i(\alpha)).$$

- Then, in case $J(t) = i$ for $t \in [U_n, U_{n+1})$, net cumulative claim process $Y(t)$ locally behaves as $Y_i(t)$:

$$Y(t) - Y(U_n) = Y_i(t) - Y_i(U_n)$$

  for all $t \in [U_n, U_{n+1})$.

With $Y_i(0) = 0$ (for all $i = 1, \ldots, d$) this mechanism fully defines $Y(t)$.

Regime switching: net cumulative claim process, ctd.

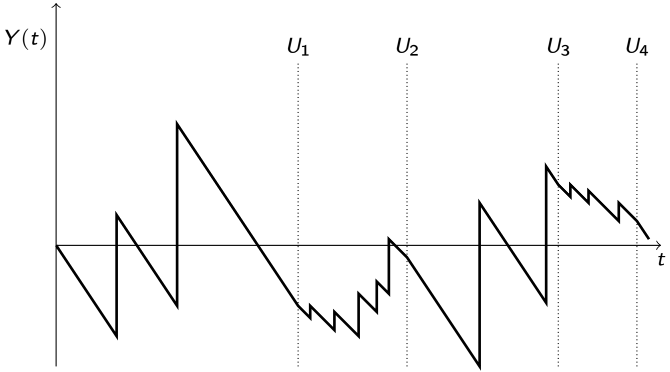

Figure: Net cumulative claim process $Y(t)$ for regime-switching compound Poisson process with $d = 2$. In this example, $J(t) = 1$ for $t \in [0, U_1)$ and $t \in [U_2, U_3)$, whereas $J(t) = 2$ for $t \in [U_1, U_2)$ and $t \in [U_3, U_4)$.

Regime switching: net cumulative claim process, ctd.

We do *not* assume that all premium rates r_i are positive.
Let S be set of indices i for which $r_i \leqslant 0$; this is set of *subordinator states*, i.e., states i for which $Y_i(t)$ is non-decreasing with probability 1.

Define

$$\bar{Y}(t) := \sup_{s \in [0,t]} Y(s),$$

$$\bar{Y}_i(t) := \sup_{s \in [0,t]} Y_i(s),$$

for $i \in \{1, \ldots, d\}$.

Goal: analyze

$$p_i(u,t) := \mathbb{P}\big(\bar{Y}(t) \geqslant u \,|\, J(0) = i\big) = \mathbb{P}\big(\bar{Z}_i(t) \geqslant u\big),$$

where $\bar{Z}_i(t)$ is $\bar{Y}(t)$ conditional on $J(0) = i$.

Regime switching: net cumulative claim process, ctd.

Goal: determine Laplace transform of $p_i(u, t)$ with respect to u, evaluated at a 'killing epoch' rather than a deterministic epoch.

Chapter 1: (exponential) killing rate was constantly β.
Now: (exponential) killing rate is β_i when $J(t) = i$.

Denote killing epoch by $\check{T}_{\boldsymbol{\beta}}$, where

$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d)^\top.$$

(As before, T_β, with scalar subscript β, still denotes exponentially distributed random variable with parameter β.)

We aim to evaluate

$$\pi_i(\alpha, \boldsymbol{\beta}) := \int_0^\infty e^{-\alpha u}\, p_i(u, \check{T}_{\boldsymbol{\beta}})\, du.$$

Non-subordinator case

Suppose $i \in \{1, \ldots, d\} \backslash S$.

Given that $J(0) = i$, time till either killing or transition of background process is exponentially distributed with parameter $\theta_i := \beta_i + q_i$ (Why?).

To exceed u, this can either happen before this epoch, or (in case event does not correspond to killing) after background process has jumped to another state. Hence,

$$p_i(u, \check{T}_{\boldsymbol{\beta}}) = \mathbb{P}\left(\bar{Y}_i(T_{\theta_i}) \geqslant u\right) + \sum_{j \neq i} \frac{q_{ij}}{\theta_i} \delta_{ij}(u),$$

with

$$\delta_{ij}(u) := \int_0^u \mathbb{P}\left(\bar{Y}_i(T_{\theta_i}) \in dv, Y_i(T_{\theta_i}) + \bar{Z}_j(\check{T}_{\boldsymbol{\beta}}) \geqslant u\right),$$

where $\bar{Z}_j(\check{T}_{\boldsymbol{\beta}})$ is independent of $(\bar{Y}_i(T_{\theta_i}), Y_i(T_{\theta_i}))$.

Non-subordinator case, ctd.

Evaluate transform (with respect to u) of first term, using results of Chapter 1.

With $\psi_i(\cdot)$ the right-inverse of $\varphi_i(\cdot)$,

$$\int_0^\infty e^{-\alpha u}\,\mathbb{P}\left(\bar{Y}_i(T_{\theta_i}) \geqslant u\right) du = \frac{1}{\alpha}\left(1 - \mathbb{E}\, e^{-\alpha \bar{Y}_i(T_{\theta_i})}\right)$$
$$= \frac{1}{\varphi_i(\alpha) - \theta_i}\left(\frac{\varphi_i(\alpha)}{\alpha} - \frac{\theta_i}{\psi_i(\theta_i)}\right).$$

Non-subordinator case, ctd.

Evaluate transform (with respect to u) of second term.
Recall: $\bar{Y}_i(T_{\theta_i})$ and $\bar{Y}_i(T_{\theta_i}) - Y_i(T_{\theta_i})$ are independent (Wiener-Hopf decomposition), with $\bar{Y}_i(T_{\theta_i}) - Y_i(T_{\theta_i})$ exponentially distributed with parameter $\chi_i := \psi_i(\theta_i)$.

Hence,

$$\int_0^\infty e^{-\alpha u}\,\delta_{ij}(u)\,du$$
$$= \int_{u=0}^\infty e^{-\alpha u} \int_{v=0}^u \int_{z=0}^\infty \mathbb{P}\left(\bar{Y}_i(T_{\theta_i}) \in dv\right)\ \chi_i e^{-\chi_i z} p_j(u - v + z, \check{T}_{\boldsymbol{\beta}})\,dz\,du$$
$$= \int_{u=0}^\infty e^{-\alpha u} \int_{v=0}^u \int_{w=u-v}^\infty \mathbb{P}\left(\bar{Y}_i(T_{\theta_i}) \in dv\right)\ \chi_i e^{-\chi_i (w-u+v)} p_j(w, \check{T}_{\boldsymbol{\beta}})\,dw\,du$$

(last step: change of variables).

Non-subordinator case, ctd.

Swap integrals:

$$\int_{v=0}^{\infty}\int_{w=0}^{\infty}\left(\int_{u=v}^{w+v} e^{-(\alpha-\chi_i)u}du\right)\mathbb{P}\left(\bar{Y}_i(T_{\theta_i})\in dv\right)\ \chi_i e^{-\chi_i(w+v)}p_j(w,\check{T}_{\boldsymbol{\beta}})\,dw.$$

Evaluating the inner integral and rearranging terms:

$$\frac{\chi_i}{\alpha-\chi_i}\int_0^{\infty} e^{-\alpha v}\mathbb{P}\left(\bar{Y}_i(T_{\theta_i})\in dv\right)\int_0^{\infty}\left(e^{-\chi_i w}-e^{-\alpha w}\right)p_j(w,\check{T}_{\boldsymbol{\beta}})\,dw.$$

Combining the above,

$$\int_0^{\infty} e^{-\alpha u}\,\delta_{ij}(u)\,du=\psi_i(\theta_i)\,\mathbb{E}\,e^{-\alpha\bar{Y}_i(T_{\theta_i})}\,\frac{\pi_j(\psi_i(\theta_i),\boldsymbol{\beta})-\pi_j(\alpha,\boldsymbol{\beta})}{\alpha-\psi_i(\theta_i)}.$$

Use expression derived for the Laplace transform of $\bar{Y}_i(T_{\theta_i})$:

$$\int_0^{\infty} e^{-\alpha u}\,\delta_{ij}(u)\,du=\theta_i\,\frac{\pi_j(\psi_i(\theta_i),\boldsymbol{\beta})-\pi_j(\alpha,\boldsymbol{\beta})}{\varphi_i(\alpha)-\theta_i}.$$

Non-subordinator case, ctd.

Upon multiplying expression of previous slide by q_{ij} and summing over $j \neq i$, following result is obtained.

Proposition

For any $\alpha \geqslant 0$ *and* $\boldsymbol{\beta} > 0$, *and* $i \in \{1, \ldots, d\} \backslash S$,

$$\pi_i(\alpha, \boldsymbol{\beta}) = \frac{1}{\varphi_i(\alpha) - \theta_i} \left(\frac{\varphi_i(\alpha)}{\alpha} - \frac{\theta_i}{\psi_i(\theta_i)} \right) + \sum_{j \neq i} q_{ij} \frac{\pi_j(\psi_i(\theta_i), \boldsymbol{\beta}) - \pi_j(\alpha, \boldsymbol{\beta})}{\varphi_i(\alpha) - \theta_i}.$$

Subordinator case

Now suppose that $i \in S$. Then $\bar{Y}_i(t) = Y_i(t)$ for any $t \geqslant 0$. Hence

$$p_i(u, \check{T}_{\boldsymbol{\beta}}) = \mathbb{P}\left(Y_i(T_{\theta_i}) \geqslant u\right) + \sum_{j \neq i} \frac{q_{ij}}{\theta_i} \eta_{ij}(u),$$

with

$$\eta_{ij}(u) := \int_0^u \mathbb{P}\left(Y_i(T_{\theta_i}) \in dv\right) \mathbb{P}\left(\bar{Z}_j(\check{T}_{\boldsymbol{\beta}}) \geqslant u - v\right).$$

First term:

$$\int_0^\infty e^{-\alpha u}\, \mathbb{P}\left(Y_i(T_{\theta_i}) \geqslant u\right) du = \frac{1}{\alpha}\left(1 - \mathbb{E}\, e^{-\alpha Y_i(T_{\theta_i})}\right) = \frac{1}{\varphi_i(\alpha) - \theta_i} \frac{\varphi_i(\alpha)}{\alpha}.$$

Second term, observing that $\eta_{ij}(u)$ is a convolution,

$$\int_0^\infty e^{-\alpha u}\, \eta_{ij}(u)\, du = \frac{\theta_i}{\theta_i - \varphi_i(\alpha)}\, \pi_j(\alpha, \boldsymbol{\beta}).$$

Subordinator case, ctd.

Proposition

For any $\alpha \geqslant 0$ *and* $\boldsymbol{\beta} > 0$, *and* $i \in S$,

$$\pi_i(\alpha, \boldsymbol{\beta}) = \frac{1}{\varphi_i(\alpha) - \theta_i} \frac{\varphi_i(\alpha)}{\alpha} - \sum_{j \neq i} q_{ij} \frac{\pi_j(\alpha, \boldsymbol{\beta})}{\varphi_i(\alpha) - \theta_i}.$$

Equations in matrix notation

Define vector of transforms

$$\boldsymbol{\pi}(\alpha,\boldsymbol{\beta}) = (\pi_1(\alpha,\boldsymbol{\beta}),\ldots,\pi_d(\alpha,\boldsymbol{\beta}))^\top.$$

Write, with $\boldsymbol{\kappa}(\alpha,\boldsymbol{\beta})$ the corresponding column vector,

$$\kappa_i(\alpha,\boldsymbol{\beta}) := \frac{\varphi_i(\alpha)}{\alpha} - \frac{\theta_i}{\psi_i(\theta_i)} 1\{i \notin S\} + \sum_{j\neq i} q_{ij}\pi_j(\psi_i(\theta_i),\boldsymbol{\beta}) 1\{i \notin S\}.$$

In addition, let (i,j)-th entry of matrix $M(\alpha,\boldsymbol{\beta})$ be given by

$$m_{ij}(\alpha,\boldsymbol{\beta}) := \big(\varphi_i(\alpha) - \theta_i\big) 1\{i = j\} + q_{ij}.$$

Proposition

For any $\alpha \geqslant 0$ and $\boldsymbol{\beta} > 0$,

$$M(\alpha,\boldsymbol{\beta})\,\boldsymbol{\pi}(\alpha,\boldsymbol{\beta}) = \boldsymbol{\kappa}(\alpha,\boldsymbol{\beta}).$$

Equations in matrix notation

Hence, for any given $\alpha > 0$ and $\boldsymbol{\beta} > 0$, if $M(\alpha, \boldsymbol{\beta})^{-1}$ exists,

$$\boldsymbol{\pi}(\alpha, \boldsymbol{\beta}) = M(\alpha, \boldsymbol{\beta})^{-1}\, \boldsymbol{\kappa}(\alpha, \boldsymbol{\beta}).$$

Denote by d° number of states in $\{1, \ldots, d\} \backslash S$.

For given vector $\boldsymbol{\beta}$ of killing rates, characterization of Proposition still contains d° unknowns:

$$\omega_i(\boldsymbol{\beta}) := -\frac{\theta_i}{\psi_i(\theta_i)} + \sum_{j \neq i} q_{ij}\, \pi_j(\psi_i(\theta_i), \boldsymbol{\beta})$$

for $i \in \{1, \ldots, d\} \backslash S$.

Next goal: identification of these d° constants.

Identification of unknown constants

Three stages: state space of $J(t)$ is

- one recurrent class,
- one transient class and one recurrent class,
- multiple transient classes and one recurrent class.

Proposition (Ivanovs–B–M)

Suppose background process $J(t)$ consists of single (hence recurrent) class. Let $Y_1(t), \ldots, Y_d(t)$ be compound Poisson processes with (not necessarily negative) drift.
Then, for any componentwise positive vector $\boldsymbol{\beta}$, equation $\det M(\alpha, \boldsymbol{\beta}) = 0$ *has d° solutions for $\alpha \in \mathbb{C}$ that have positive real part.*

Identification of unknown constants, ctd.

Start with case of one recurrent class.
Define matrix $M_{\boldsymbol{\kappa},i}(\alpha, \boldsymbol{\beta})$ as matrix $M(\alpha, \boldsymbol{\beta})$ but with i-th column replaced by $\boldsymbol{\kappa}(\alpha, \boldsymbol{\beta})$.

Then, by $M(\alpha, \boldsymbol{\beta})\, \boldsymbol{\pi}(\alpha, \boldsymbol{\beta}) = \boldsymbol{\kappa}(\alpha, \boldsymbol{\beta})$ and Cramer's rule,

$$\pi_i(\alpha, \boldsymbol{\beta}) = \frac{det\, M_{\boldsymbol{\kappa},i}(\alpha, \boldsymbol{\beta})}{det\, M(\alpha, \boldsymbol{\beta})}.$$

As $\pi_i(\alpha, \boldsymbol{\beta})$ is finite, any zero of denominator should be zero of the numerator. Because $J(t)$ is irreducible, we can apply Proposition: $det\, M(\alpha, \boldsymbol{\beta}) = 0$ has d° zeroes in right half of complex plane.

Assume that these zeroes have multiplicity 1; we call them $\alpha_1, \ldots, \alpha_{d^\circ}$ (each of them depending on vector of killing rates $\boldsymbol{\beta}$).

Identification of unknown constants, ctd.

For given $\boldsymbol{\beta}$ and $i = 1, \ldots, d$ and $j = 1, \ldots, d^\circ$,

$$det\, M_{\boldsymbol{\kappa},i}(\alpha_j, \boldsymbol{\beta}) = 0.$$

This seemingly yields $d \times d^\circ$ equations to determine the d° unknowns ω_i (for $i \notin S$). However, all equations that correspond to specific index $j \in \{1, \ldots, d^\circ\}$ effectively provide same information.

This is shown as follows.

Identification of unknown constants, ctd.

Let $\boldsymbol{m}_i(\alpha, \boldsymbol{\beta})$ be i-th column of $M(\alpha, \boldsymbol{\beta})$. Suppose (for fixed i) that $det\, M(\alpha, \boldsymbol{\beta}) = 0$ and $det\, M_{\boldsymbol{\kappa},i}(\alpha, \boldsymbol{\beta}) = 0$ for some $\alpha \in \mathbb{C}$ with a positive real part. Hence $M(\alpha, \boldsymbol{\beta})$ and $M_{\boldsymbol{\kappa},i}(\alpha, \boldsymbol{\beta})$ are singular, so that

$$\sum_{j=1}^{d} \boldsymbol{m}_j(\alpha, \boldsymbol{\beta})\, v_j = 0, \quad \sum_{j \neq i} \boldsymbol{m}_j(\alpha, \boldsymbol{\beta})\, u_j + \boldsymbol{\kappa}(\alpha, \boldsymbol{\beta})\, u_i = 0$$

for some $\boldsymbol{u}$ and $\boldsymbol{v}$. Therefore, for any $i' \neq i$,

$$\begin{aligned} 0 &= -u_{i'} \sum_{j=1}^{d} \boldsymbol{m}_j(\alpha, \boldsymbol{\beta})\, v_j + v_{i'} \sum_{j \neq i} \boldsymbol{m}_j(\alpha, \boldsymbol{\beta})\, u_j + v_{i'}\, \boldsymbol{\kappa}(\alpha, \boldsymbol{\beta})\, u_i \\ &= -u_{i'} v_i \boldsymbol{m}_i(\alpha, \boldsymbol{\beta}) + \sum_{j \neq i, i'} (v_{i'} u_j - u_{i'} v_j)\, \boldsymbol{m}_j(\alpha, \boldsymbol{\beta}) + v_{i'} u_i\, \boldsymbol{\kappa}(\alpha, \boldsymbol{\beta}). \end{aligned}$$

We found linear combination of columns of $M_{\boldsymbol{\kappa},i'}(\alpha, \boldsymbol{\beta})$ that equals 0. Hence, $M_{\boldsymbol{\kappa},i'}(\alpha, \boldsymbol{\beta})$ is singular, and $det\, M_{\boldsymbol{\kappa},i'}(\alpha, \boldsymbol{\beta}) = 0$. Conclude that, for j fixed, varying i does not provide any additional constraints.

Identification of unknown constants, ctd.

Hence, for any $j = 1, \dots, d^\circ$ we can focus on $det\, M_{\boldsymbol{\kappa},1}(\alpha_j, \boldsymbol{\beta}) = 0$ only (we take $i = 1$, that is).

Let $\bar{M}_{ij}(\alpha, \boldsymbol{\beta})$ represent $(d-1) \times (d-1)$ matrix which results after deleting i-th row and j-th column from $M(\alpha, \boldsymbol{\beta})$. Recall that

$$\kappa_i(\alpha, \boldsymbol{\beta}) = \frac{\varphi_i(\alpha)}{\alpha} + \omega_i(\boldsymbol{\beta}) 1\{i \notin S\},$$

the equation $det\, M_{\boldsymbol{\kappa},1}(\alpha_j, \boldsymbol{\beta}) = 0$ can be rewritten as

$$\sum_{i \in S} \frac{\varphi_i(\alpha_j)}{\alpha_j} (-1)^{1+i} det\, \bar{M}_{i1}(\alpha_j, \boldsymbol{\beta}) \\ + \sum_{i \notin S} \left(\frac{\varphi_i(\alpha_j)}{\alpha_j} + \omega_i(\boldsymbol{\beta}) \right) (-1)^{1+i} det\, \bar{M}_{i1}(\alpha_j, \boldsymbol{\beta}) = 0.$$

We thus obtain d° equations (linear in d° unknowns $\omega_1(\boldsymbol{\beta}), \dots, \omega_{d^\circ}(\boldsymbol{\beta})$).

Identification of unknown constants, ctd.

Now: single transient class, say $T \subset \{1, \ldots, d\}$, besides the recurrent states (which could correspond to single class or multiple classes).

We know how to compute $\pi_i(\alpha, \boldsymbol{\beta})$ for any $i \notin T$. For $i \in T$,

$$\sum_{j \in T} m_{ij}(\alpha, \boldsymbol{\beta})\, \pi_j(\alpha_j, \boldsymbol{\beta}) = \kappa_i(\alpha, \boldsymbol{\beta}) - \sum_{j \notin T} m_{ij}(\alpha, \boldsymbol{\beta})\, \pi_j(\alpha, \boldsymbol{\beta}).$$

Right-hand side we know; denote it by $\bar{\kappa}_i(\alpha, \boldsymbol{\beta})$. Define $\bar{d} := |T|$ and $\bar{d}^\circ := |T \backslash S|$. In addition, define $\bar{d} \times \bar{d}$ matrix

$$\bar{M}(\alpha, \boldsymbol{\beta}) := (m_{ij}(\alpha, \boldsymbol{\beta}))_{i,j \in T},$$

and let $\bar{d}$-dimensional vector $\bar{\boldsymbol{\pi}}(\alpha, \boldsymbol{\beta})$ represent the entries of $\boldsymbol{\pi}(\alpha, \boldsymbol{\beta})$ that correspond to states in T. As a result, we have found the equation

$$\bar{M}(\alpha, \boldsymbol{\beta})\, \bar{\boldsymbol{\pi}}(\alpha, \boldsymbol{\beta}) = \bar{\boldsymbol{\kappa}}(\alpha, \boldsymbol{\beta}).$$

Identification of unknown constants, ctd.

Suppose det $\bar{M}(\alpha, \boldsymbol{\beta}) = 0$ has $\bar{d}^\circ$ zeroes in the right half of the complex plane, then we would be done.

In light of Proposition, we have to verify that entries of $\bar{M}(\alpha, \boldsymbol{\beta})$ are of the form

$$\bar{m}_{ij}(\alpha, \boldsymbol{\beta}) := \big(\varphi_i(\alpha) - \theta_i\big)1\{i = j\} + \bar{q}_{ij},$$

with transition rates $\bar{q}_{ij}$ corresponding to *single* recurrent class.

Rewrite diagonal elements of $\bar{M}(\alpha, \boldsymbol{\beta})$ by adapting diagonal elements of rate matrix and killing rates:

$$m_{ii}(\alpha, \boldsymbol{\beta}) = \varphi_i(\alpha) - \beta_i + q_{ii} = \varphi_i(\alpha) - \bar{\beta}_i + \bar{q}_{ii},$$

with

$$\bar{q}_{ii} := -\sum_{j \in T \setminus \{i\}} q_{ij}, \quad \bar{\beta}_i := \left(\beta_i + \sum_{j \notin T} q_{ij}\right).$$

Conclude: $\bar{M}(\alpha, \boldsymbol{\beta})$ has desired form.

Identification of unknown constants, ctd.

Hence, Proposition applies, implying that $\bar{M}(\alpha, \boldsymbol{\beta}) = 0$ indeed has $\bar{d}^{\circ}$ zeroes in right half of complex plane (for any componentwise positive vector $\boldsymbol{\beta}$).

We can therefore identify $\omega_i(\boldsymbol{\beta})$ for $i \in T \backslash S$ by solving linear system, as before.

Identification of unknown constants, ctd.

Finally: multiple transient classes, say $T_1, \ldots, T_K$. Let R denote union of all recurrent states.

Write $T_k \rightsquigarrow T_{k'}$, with $k, k' \in \{1, \ldots, K\}$, if there is a *direct* transition from a state in T_k to a state in $T_{k'}$ (i.e., if there are $i \in T_k$ and $j \in T_{k'}$ such that $q_{ij} > 0$).

Define 'layers' recursively: $C_0 := R$, and

$$C_n := \left\{ T_k : \text{ for all } k' \text{ such that } T_k \rightsquigarrow T_{k'} \text{ it holds that } k' \in \bigcup_{m=0}^{n-1} C_m \right\}.$$

Observe: number of layer sets C_n is (including C_0) at most K.

Identification of unknown constants, ctd.

Above: computation of $\pi_i(\alpha, \boldsymbol{\beta})$ for $i \in R$ and $i \in C_1$. Now: $\pi_i(\alpha, \boldsymbol{\beta})$ for $i \in C_n$, knowing $\pi_i(\alpha, \boldsymbol{\beta})$ for $i \in R, C_1, \ldots, C_{n-1}$.

Suppose $T_k \subseteq C_n$. As states in C_n have no direct transitions to classes outside C_{n-1}, we have for $i \in T_k$ that

$$\sum_{j \in T_k} m_{ij}(\alpha, \boldsymbol{\beta})\, \pi_j(\alpha, \boldsymbol{\beta}) = \kappa_i(\alpha, \boldsymbol{\beta}) - \sum_{j \in C_{n-\mathbf{1}}} m_{ij}(\alpha, \boldsymbol{\beta})\, \pi_j(\alpha, \boldsymbol{\beta}).$$

As right-hand side contains known quantities only, analysis is as in case of single transient class. Specifically, number of zeroes (in right half of complex plane) of determinant of $(m_{ij}(\alpha, \boldsymbol{\beta}))_{i,j \in T_k}$ equals number of states in T_k that do not correspond to non-decreasing subordinators.

CL over phase-type horizon

Chapter 1: conventional CL model, with focus on double transform of $p(u,t)$. Transform over time: ruin over *exponentially distributed* interval. Now: extension to class of *phase-type intervals* $\mathscr{P}$.

Class $\mathscr{P}$ is relevant, as any distribution on the positive half-line can be approximated arbitrarily closely by distribution in $\mathscr{P}$.

This actually holds true for the smaller class $\mathscr{P}^\circ \subset \mathscr{P}$ of mixtures of Erlang distributions.

CL over phase-type horizon, ctd.

Phase-type distribution: absorption time of a continuous-time Markov chain. That is, each distribution in $\mathscr{P}$ is characterized by

- a finite state space $\{1, ..., d\}$,
- initial probability vector $\boldsymbol{\delta} \in \mathbb{R}^d$,
- $d \times d$ transition rate matrix $F = (f_{ij})_{i,j=1}^{d}$ (i.e., it has non-positive diagonal elements, non-negative non-diagonal elements, and row sums equal to zero),
- non-negative *exit vector* $\boldsymbol{f}$.

CL over phase-type horizon, ctd.

We define additional transition rate matrix, with diag($\boldsymbol{f}$) denoting diagonal matrix with $\boldsymbol{f}$ on its diagonal,

$$\bar{F} := \begin{pmatrix} F - \text{diag}(\boldsymbol{f}) & \boldsymbol{f} \\ \boldsymbol{0}^\top & 0 \end{pmatrix}.$$

Dimension of $\bar{F}$ is $(d+1) \times (d+1)$, where state $d+1$ is *absorbing state*. Note that $\bar{F}$ is genuine transition rate matrix: row sums equal 0.

Definition of *phase-type random variable*: time it takes to reach absorbing state, if initial state has been drawn according to distribution $\boldsymbol{\delta}$.
Rule out matrices $\bar{F}$ in which, starting from any state i with $\delta_i > 0$, state $d+1$ is not eventually reached.

CL over phase-type horizon, ctd.

Now consider compound Poisson process with negative drift, say $Y(t)$.
Consider $P \in \mathscr{P}$ characterized by parameters $(d, \boldsymbol{\delta}, F, \boldsymbol{f})$.

Objective: evaluate

$$\int_0^\infty \int_0^\infty e^{-\alpha u} p(u,t)\, \mathbb{P}(P \in dt)\, du.$$

Evaluation of this transform falls in our framework:

- let $Y_1(t), \ldots, Y_d(t)$ be independent copies of $Y(t)$, such that compound Poisson process with drift is same for any state of background process (say with Laplace exponent $\varphi(\alpha)$),
- to represent the killed state, let $Y_{d+1}(t) \equiv 0$,
- choose $Q = F$ and $\boldsymbol{\beta} = \boldsymbol{f}$ such that absorption in state $d+1$ corresponds to killing.

Immediate: above transform equals $\sum_{i=1}^d \delta_i \pi_i(\alpha, \boldsymbol{\beta})$.

CL over phase-type horizon, ctd.

Any distribution on the positive half-line can be approximated arbitrarily closely by distribution in $\mathscr{P}^\circ$, i.e., the class of mixtures of Erlang distributions.

Let $\boldsymbol{\delta} \equiv (\delta_1, \dots, \delta_d)$ be probability vector, and $E_k(\beta)$ be Erlang distributed rv with parameters $k \in \mathbb{N}$ and $\beta > 0$. This means

$$\mathbb{P}(E_k(\beta) \in dt) = e^{-\beta t} \frac{\beta^k t^{k-1}}{(k-1)!} dt.$$

Then $P \in \mathscr{P}^\circ$ is characterized by $\boldsymbol{\delta}$, $\boldsymbol{\beta} \in \mathbb{R}_+^d$:

$$\mathbb{P}(P \in dt) = \sum_{i=1}^{d} \delta_i \, \mathbb{P}\big(E_{k_i}(\beta_i) \in dt\big).$$

CL over phase-type horizon, ctd.

Hence, to evaluate, for $P \in \mathscr{P}^\circ$,

$$\int_0^\infty \int_0^\infty e^{-\alpha u} p(u,t)\, \mathbb{P}(P \in dt)\, du,$$

it suffices to be able to evaluate it for an $E_k(\beta)$-distributed horizon.

Indeed, if we can compute

$$\begin{aligned}\pi^{[k]}(\alpha,\beta) &:= \int_0^\infty \int_0^\infty e^{-\alpha u} p(u,t)\, \mathbb{P}(E_k(\beta) \in dt)\, du \\ &= \int_0^\infty \int_0^\infty e^{-\alpha u} p(u,t)\, e^{-\beta t} \frac{\beta^k t^{k-1}}{(k-1)!}\, dt\, du,\end{aligned}$$

then transform can be expressed as

$$\sum_{i=1}^d \delta_i\, \pi^{[k_i]}(\alpha,\beta_i).$$

CL over phase-type horizon, ctd.

But $\pi^{[k]}(\alpha,\beta)$ can be computed easily from $\pi(\alpha,\beta) \equiv \pi^{[1]}(\alpha,\beta)$, i.e., the transform corresponding to the ruin probability over *exponentially distributed* horizon.
To this end, define

$$\pi^{(\ell)}(\alpha,\beta) := \frac{d^\ell}{d\beta^\ell}\pi(\alpha,\beta).$$

Proposition

For $k \in \mathbb{N}$,

$$\pi^{[k]}(\alpha,\beta) = \sum_{\ell=0}^{k-1} \frac{(-\beta)^\ell}{\ell!}\pi^{(\ell)}(\alpha,\beta).$$

CL over phase-type horizon, ctd.

Proof. Definition of $\pi^{[k]}(\alpha,\beta)$ implies that

$$\pi^{[k]}(\alpha,\beta) = -\frac{(-\beta)^k}{(k-1)!}\left(\frac{d^{k-1}}{d\beta^{k-1}}\frac{\pi(\alpha,\beta)}{\beta}\right).$$

Observing that, by the binomium,

$$\frac{d^{k-1}}{d\beta^{k-1}}\frac{\pi(\alpha,\beta)}{\beta} = -\sum_{\ell=0}^{k-1}\binom{k-1}{\ell}\pi^{(\ell)}(\alpha,\beta)\frac{(k-1-\ell)!}{(-\beta)^{k-\ell}},$$

the stated follows immediately.

Resampling

At Poisson instants (rate ν) the 'parameters' of the net cumulative claim process become $(\lambda_i, b_i(\alpha), r_i)$ with probability p_i, where $i = 1, \ldots, d$, independent of history.

Motivation: every now and then, environment randomly changes, modeled by *resampling*.

This fits in model of this chapter, when picking

$$Q = \nu\, 1\boldsymbol{p}^\top - \nu I_d,$$

with 1 an all-ones vector and I_d the d-dimensional identity matrix. (Check!)

Chapter 3: concluding remarks

In Exercise 3.1 you will consider a modulating process with one transient and one recurrent state.

In Exercise 3.2 you will consider a two-dimensional modulating process with one states corresponding to a non-decreasing subordinator.

CHAPTER IV: INTEREST AND TWO-SIDED JUMPS

Interest & two-sided jumps: main ideas

Compared to conventional CL analysis, three additional elements are introduced:

- Insurance firm receives dividend over its reserve level. We apply interest rate $r^\circ \geqslant 0$.
- Besides claims, leading to negative jumps of reserve level, we also allow positive jumps (to be thought of as capital injections).
- As before we aim at characterizing probability of ruin (transformed with respect to the initial capital surplus level) before exponentially distributed time, but now jointly with three other quantities: time of ruin, undershoot, and overshoot. See also Exercise 1.2.

Interest & two-sided jumps: main ideas, ctd.

Objective: analyze, for a given initial capital surplus level u,

$$p(u,t,\boldsymbol{\gamma}) := \mathbb{E}\Big(e^{-\gamma_1\tau(u)-\gamma_2 X_u(\tau(u)-)-\gamma_3 X_u(\tau(u))}1\{\tau(u)\leqslant t\}\Big),$$

where $\boldsymbol{\gamma}\equiv(\gamma_1,\gamma_2,\gamma_3)^\top$, by evaluating its transform.

With T_β exponentially distributed time with parameter β, we consider

$$\pi(\alpha,\beta,\boldsymbol{\gamma}) := \int_0^\infty e^{-\alpha u}\beta e^{-\beta t}p(u,t,\boldsymbol{\gamma})\,dt\,du = \int_0^\infty e^{-\alpha u}p(u,T_\beta,\boldsymbol{\gamma})\,du.$$

Plugging in $\boldsymbol{\gamma}=0$, we recover objects of Chapter 1.

For conciseness, in the sequel we write, for given $\beta>0$ and $\boldsymbol{\gamma}$ such that $\gamma_1,\gamma_2\geqslant 0$ and $\gamma_3\leqslant 0$ (Why these signs?),

$$p(u)\equiv p(u,T_\beta,\boldsymbol{\gamma}) = \mathbb{E}\Big(e^{-\gamma_1\tau(u)-\gamma_2 X_u(\tau(u)-)-\gamma_3 X_u(\tau(u))}1\{\tau(u)\leqslant T_\beta\}\Big).$$

Interest & two-sided jumps: main ideas, ctd.

First model extension:

- As before, claims arriving according to Poisson process, corresponding to *downward* jumps in $X_u(t)$. Here $\lambda_- \geqslant 0$ is arrival rate, and $b_-(\alpha)$ the LST of generic claim size B_-.
- In addition, there are *upward* jumps in $X_u(t)$, which could for instance represent capital injections, arriving according to Poisson process with rate $\lambda_+ \geqslant 0$. We let $b_+(\alpha)$ be LST of generic upward jump B_+.

Interest & two-sided jumps: main ideas, ctd.

Second model extension: insurance company receives *interest* (at rate $r^\circ \geqslant 0$) over its current reserve level.

Hence, with S_i denoting i-th jump epoch of the reserve level process $X_u(t)$, between two consecutive jump epochs S_i and S_{i+1} process $X_u(t)$ evolves according to differential equation

$$dX_u(t) = r\,dt + r^\circ X_u(t)\,dt.$$

It follows that, for $t \in (S_i, S_{i+1})$,

$$X_u(t) = X_u(S_i)\,e^{r^\circ(t-S_i)} + \frac{r}{r^\circ}\left(e^{r^\circ(t-S_i)} - 1\right).$$

Interest & two-sided jumps: main ideas, ctd.

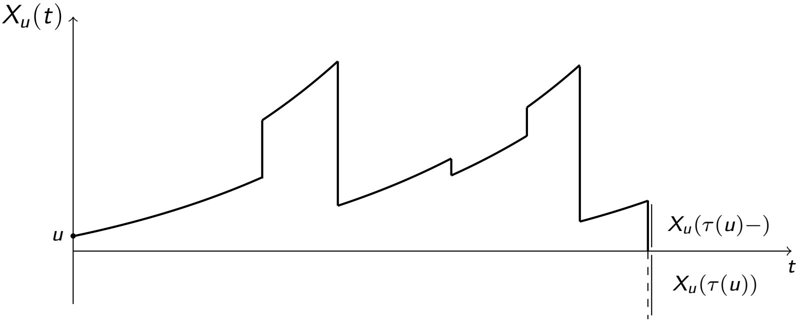

Figure: Sample path of $X_u(t)$ until $\tau(u)$. Upward jumps are distributed as generic random variable B_+, downward jumps are distributed as generic random variable B_-.

Exponential upward jumps

First step: by 'classical Markovian reasoning', with $\lambda := \lambda_- + \lambda_+$,

$$\begin{aligned} p(u) = e^{-\gamma_1 \Delta t} \Big(& \lambda_- \Delta t \int_0^u \mathbb{P}(B_- \in dv)\, p(u-v) \\ & + \lambda_- \Delta t \int_u^\infty \mathbb{P}(B_- \in dv)\, e^{-\gamma_2 u}\, e^{-\gamma_3 (u-v)} \\ & + \lambda_+ \Delta t \int_0^\infty \mu e^{-\mu v} p(u+v)\, dv \\ & + (1 - \lambda\,\Delta t - \beta\,\Delta t)\, p(u + r\,\Delta t + r^\circ u\,\Delta t) \Big) + o(\Delta t). \end{aligned}$$

- Use that between jumps process grows according to solution of differential equation.
- In considered interval of length Δt time till ruin $\tau(u)$ grows by Δt.
- Undershoot $X_u(\tau(u)-)$ and overshoot $X_u(\tau(u))$ can be assigned their values when surplus level drops below 0 (due to negative jump of size at least u).

Exponential upward jumps, ctd.

Linearize $e^{-\gamma_1 \Delta t}$ and $p(u + r\,\Delta t + r^\circ u\,\Delta t)$: as $\Delta t \downarrow 0$,

$$p(u) = p(u + r\,\Delta t + r^\circ u\,\Delta t) + \lambda_-\,\Delta t \int_0^u \mathbb{P}(B_- \in dv)\, p(u-v)$$
$$+ \lambda_-\,\Delta t \int_u^\infty \mathbb{P}(B_- \in dv)\, e^{-\gamma_2 u}\, e^{-\gamma_3 (u-v)}$$
$$+ \lambda_+\,\Delta t \int_0^\infty \mu e^{-\mu v} p(u+v)\, dv - (\gamma_1 + \lambda + \beta)\,\Delta t\, p(u) + o(\Delta t).$$

Subtract $p(u + r\,\Delta t + r^\circ u\,\Delta t)$, and divide by Δt: as $\Delta t \downarrow 0$,

$$-\frac{p(u + r\,\Delta t + r^\circ u\,\Delta t) - p(u)}{r\,\Delta t + r^\circ u\,\Delta t}\,(r + r^\circ u) = \lambda_- \int_0^u \mathbb{P}(B_- \in dv)\, p(u-v)$$
$$+ \lambda_- \int_u^\infty \mathbb{P}(B_- \in dv)\, e^{-\gamma_2 u}\, e^{-\gamma_3 (u-v)}$$
$$+ \lambda_+ \int_0^\infty \mu e^{-\mu v} p(u+v)\, dv - (\gamma_1 + \lambda + \beta)\, p(u) + o(1).$$

Exponential upward jumps, ctd.

Then take limit $\Delta t \downarrow 0$, to obtain following integro-differential equation.

Lemma

For any $u > 0$,

$$-p'(u)\,(r + r^\circ u) = \lambda_- \int_0^u \mathbb{P}(B_- \in dv) p(u - v) + \lambda_- \int_u^\infty \mathbb{P}(B_- \in dv)\, e^{-\gamma_2 u}\, e^{-\gamma_3 (u-v)} + \lambda_+ \int_0^\infty \mu e^{-\mu v} p(u + v)\, dv - (\gamma_1 + \lambda + \beta)\, p(u).$$

Exponential upward jumps, ctd.

Next goal is to evaluate $\bar{\pi}(\alpha) \equiv \bar{\pi}(\alpha, \beta, \boldsymbol{\gamma}) := \alpha\pi(\alpha, \beta, \boldsymbol{\gamma})$ (interpretation: $p(u)$ in which the initial reserve level u is exponentially distributed with parameter α).

Transform full integro-differential equation of Lemma with respect to u: multiply both sides by $\alpha e^{-\alpha u}$, and integrate over $u \in (0, \infty)$.

Objective: obtain equation that is fully expressed in terms of $\bar{\pi}(\alpha)$. We do so by considering each term separately.

Exponential upward jumps, ctd.

- First term LHS: by integration by parts,

$$-\int_0^\infty p'(u)\, r\, \alpha e^{-\alpha u}\, du = r\alpha\big(p(0) - \bar{\pi}(\alpha)\big).$$

- Second term LHS:

$$\begin{aligned}-\int_0^\infty p'(u)\, r^\circ u\, \alpha e^{-\alpha u}\, du &= r^\circ \alpha \int_0^\infty p(u)\big(e^{-\alpha u} - u\, \alpha e^{-\alpha u}\big)\, du \\ &= r^\circ \alpha \bar{\pi}'(\alpha),\end{aligned}$$

using standard identity

$$\bar{\pi}'(\alpha) = \frac{\bar{\pi}(\alpha)}{\alpha} - \int_0^\infty u\, \alpha e^{-\alpha u} p(u)\, du.$$

Exponential upward jumps, ctd.

- First term RHS: upon interchanging the order of the integrals,

$$\lambda_- \int_0^\infty \left(\int_0^u \mathbb{P}(B_- \in dv)\, p(u-v)\, dv \right) \alpha e^{-\alpha u}\, du$$
$$= \lambda_- \int_0^\infty e^{-\alpha v} \left(\int_v^\infty p(u-v)\, \alpha e^{-\alpha(u-v)} du \right) \mathbb{P}(B_- \in dv)$$
$$= \lambda_- b_-(\alpha) \bar{\pi}(\alpha).$$

- Second term RHS:

$$\lambda_- \int_0^\infty \left(\int_u^\infty \mathbb{P}(B_- \in dv) e^{-\gamma_2 u}\, e^{-\gamma_3 (u-v)} \right) \alpha e^{-\alpha u}\, du$$
$$= \lambda_- \alpha \int_0^\infty \frac{e^{\gamma_3 v} - e^{-(\alpha+\gamma_2)v}}{\alpha+\gamma_2+\gamma_3} \mathbb{P}(B_- \in dv) = \lambda_- \alpha \frac{b_-(-\gamma_3) - b_-(\alpha+\gamma_2)}{\alpha+\gamma_2+\gamma_3}.$$

Note: $\alpha = -\gamma_2 - \gamma_3$ is a removable singularity (Why?).

Exponential upward jumps, ctd.

- Third term RHS: applying transformation $w := u + v$,

$$\lambda_+ \int_0^\infty \Big(\int_0^\infty \mu e^{-\mu v} p(u+v)\, dv \Big)\, \alpha e^{-\alpha u}\, du$$
$$= \lambda_+ \frac{\mu}{\mu - \alpha} \bar{\pi}(\alpha) - \lambda_+ \frac{\alpha}{\mu - \alpha} \bar{\pi}(\mu).$$

  Notice: $\alpha = \mu$ is removable singularity, but requires some extra care.

- Fourth term RHS: by the definition of $\bar{\pi}(\alpha)$,

$$-\int_0^\infty (\gamma_1 + \lambda + \beta)\, p(u)\, \alpha e^{-\alpha u}\, du = -(\gamma_1 + \lambda + \beta)\, \bar{\pi}(\alpha).$$

Exponential upward jumps, ctd.

Introduce some notation: we let $A := -(\gamma_1 + \beta)/r^\circ$ and

$$F(\alpha) := \bar{F}(\alpha) + \frac{A}{\alpha}, \qquad \bar{F}(\alpha) := \frac{r}{r^\circ} - \frac{\lambda_-}{r^\circ}\frac{1 - b_-(\alpha)}{\alpha} + \frac{\lambda_+}{r^\circ}\frac{1}{\mu - \alpha},$$
$$G(\alpha) := \frac{\lambda_-}{r^\circ}\frac{b_-(-\gamma_3) - b_-(\alpha + \gamma_2)}{\alpha + \gamma_2 + \gamma_3} - \frac{r}{r^\circ}p(0) - \frac{\lambda_+}{r^\circ}\frac{1}{\mu - \alpha}\bar{\pi}(\mu).$$

Proposition

For any $\alpha \geqslant 0$, $\bar{\pi}(\cdot)$ fulfils the differential equation

$$\bar{\pi}'(\alpha) = F(\alpha)\,\bar{\pi}(\alpha) + G(\alpha).$$

Exponential upward jumps, ctd.

Differential equation of Proposition is routinely solved using the method of variation of constants. With $F_\star(\alpha)$ the primitive of $F(\alpha)$:

$$\bar{\pi}(\alpha) = \left(\int_0^\alpha G(\eta)\exp\big(-F_\star(\eta)\big)\,d\eta + K\right)\exp\big(F_\star(\alpha)\big).$$

As a consequence of the fact that $F_\star(\alpha)\to\infty$ as $\alpha\to\infty$ (Check!), we have that $\bar{\pi}(\infty) = p(0) \in (0,1)$ necessarily implies that

$$K = -\int_0^\infty G(\eta)\exp\big(-F_\star(\eta)\big)\,d\eta.$$

Hence,

$$\bar{\pi}(\alpha) = -\left(\int_\alpha^\infty G(\eta)\exp\big(-F_\star(\eta)\big)\,d\eta\right)\exp\big(F_\star(\alpha)\big).$$

Exponential upward jumps, ctd.

Left: determination of the two unknown constants $p(0)$ and $\bar{\pi}(\mu)$. To identify these, write $G(\alpha) = p(0)\, G_1(\alpha) + \bar{\pi}(\mu)\, G_2(\alpha) + G_3(\alpha)$, where

$$G_1(\alpha) := -\frac{r}{r^\circ}, \quad G_2(\alpha) := -\frac{\lambda_+}{r^\circ}\frac{1}{\mu-\alpha}, \quad G_3(\alpha) := \frac{\lambda_-}{r^\circ}\frac{b_-(-\gamma_3) - b_-(\alpha+\gamma_2)}{\alpha+\gamma_2+\gamma_3}.$$

Analogously, define $I(\alpha)$ as $p(0)\, I_1(\alpha) + \bar{\pi}(\mu)\, I_2(\alpha) + I_3(\alpha)$, where

$$I_k(\alpha) := \int_\alpha^\infty G_k(\eta) \exp\big(-F_\star(\eta)\big)\, d\eta.$$

To obtain constraints that are used to determine $p(0)$ and $\bar{\pi}(\mu)$, note that if for some α we have that $F_\star(\alpha) = \infty$, then necessarily $I(\alpha) = 0$, due to finiteness of $\bar{\pi}(\alpha)$.

Exponential upward jumps, ctd.

- Shape of $F(\cdot)$ reveals that, for some constant $D_0 < 0$,

$$\lim_{\alpha \downarrow 0} \frac{F_\star(\alpha)}{\log \alpha} = D_0,$$

which implies that $F_\star(\alpha) \to \infty$ as $\alpha \downarrow 0$, and hence $I(0) = 0$ (so that $K = 0$). We find

$$p(0) I_1(0) + \bar{\pi}(\mu) I_2(0) = -I_3(0).$$

- Analogously, for some constants $\bar{D}_\mu \in \mathbb{R}$ and $D_\mu < 0$,

$$\lim_{\alpha \uparrow \mu} \frac{F_\star(\alpha) - \bar{D}_\mu}{\log(\mu - \alpha)} = D_\mu,$$

so that $F_\star(\alpha) \to \infty$ as $\alpha \uparrow \mu$. Hence, $I(\mu) = 0$, and therefore

$$p(0) I_1(\mu) + \bar{\pi}(\mu) I_2(\mu) = -I_3(\mu).$$

Exponential upward jumps, ctd.

Hence: two linear equations, in equally many unknowns. We find

$$p(0) = -\frac{I_3(0)I_2(\mu) - I_3(\mu)I_2(0)}{I_1(0)I_2(\mu) - I_1(\mu)I_2(0)}, \quad \bar{\pi}(\mu) = -\frac{I_1(0)I_3(\mu) - I_1(\mu)I_3(0)}{I_1(0)I_2(\mu) - I_1(\mu)I_2(0)}.$$

We have thus arrived at final result.

Theorem

If $r > 0$, then

$$\bar{\pi}(\alpha) = -\left(\int_\alpha^\infty G(\eta)\exp\big(-F_\star(\eta)\big)\,d\eta\right)\exp\big(F_\star(\alpha)\big),$$

with $p(0)$ and $\bar{\pi}(\mu)$ given above.

Exponential upward jumps, ctd.

Next result: alternative way to describe $\bar{\pi}(\cdot)$: through a power series expansion.

Writing, for coefficients $\bar{f}_\ell$ and g_ℓ,

$$\bar{F}(\alpha) = \sum_{\ell=0}^{\infty} \bar{f}_\ell \alpha^\ell, \quad G(\alpha) = \sum_{\ell=0}^{\infty} g_\ell \alpha^\ell,$$

we have found differential equation

$$\bar{\pi}'(\alpha) = \left(\sum_{\ell=0}^{\infty} \bar{f}_\ell \alpha^\ell + \frac{A}{\alpha} \right) \bar{\pi}(\alpha) + \sum_{\ell=0}^{\infty} g_\ell \alpha^\ell.$$

Writing $c_\ell := \bar{\pi}^{(\ell)}(0)$, this differential equation can be rewritten to

$$\sum_{\ell=0}^{\infty} \frac{c_{\ell+1}}{\ell!} \alpha^\ell = \left(\sum_{\ell=0}^{\infty} \bar{f}_\ell \alpha^\ell + \frac{A}{\alpha} \right) \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \alpha^\ell + \sum_{\ell=0}^{\infty} g_\ell \alpha^\ell.$$

Exponential upward jumps, ctd.

Collect terms corresponding to same power in both sides, coefficients c_k can be determined. After some algebra, we find that c_k obey following recursion.

Proposition

The power series expansion of $\bar{\pi}(\alpha)$ *is* $\sum_{\ell=0}^{\infty} c_\ell \, \alpha^\ell/\ell!$*, where* $c_0 = 0$ *and, for* $\ell \in \mathbb{N}$*,*

$$c_{\ell+1} = \left(\frac{1}{\ell!} - \frac{A}{(\ell+1)!}\right)^{-1} \left(\sum_{m=0}^{\ell} \bar{f}_m c_{\ell-m} + g_\ell\right).$$

Relaxation of exponentiality assumptions

Seeming drawback: exponentiality assumptions imposed.

Concretely, $\bar{\pi}(\alpha)$ corresponds to the situation in which

- initial reserve level,
- killing, and
- upward jumps

are assumed exponentially distributed.

What can we do about this?

Relaxation of exponentiality assumptions, ctd.

Section 3.4: approximate distribution on the positive half-line by a distribution in the class of phase-type distributions $\mathscr{P}$. Even smaller class of distributions suffices: $\mathscr{P}^\circ$, class of mixtures of Erlang distributions.

For instance: any number $z > 0$ can be approximated arbitrarily closely by Erlang distribution with shape parameter k and scale parameter k/z, with k large.

Goal: compute $p(U, T, \gamma)$ with initial level U and time horizon T in $\mathscr{P}^\circ$. Extends results of Section 4.3, where we found $\bar{\pi}(\alpha) = p(U_\alpha, T_\beta, \gamma)$, with U_α exponentially distributed rv with mean α^{-1}.

Section 3.4: to deal with distributions in $\mathscr{P}^\circ$, it suffices to deal with U and T Erlang distributed. Relying on Proposition 3.5, translate results for U or T being exponentially distributed to their Erlang counterpart. Following example presents explicit procedure, for Erlang distributed initial reserve level U.

Relaxation of exponentiality assumptions, ctd.

Let initial level U be Erlang with parameters k and α.

Idea: use Proposition 3.5. Requires derivatives $\bar{\pi}^{(\ell)}(\cdot)$. We know $\bar{\pi}(\alpha)$, so our differential equation gives

$$\bar{\pi}^{(1)}(\alpha) = F(\alpha)\bar{\pi}(\alpha) + G(\alpha).$$

But then also

$$\begin{aligned}\bar{\pi}^{(2)}(\alpha) &= F^{(1)}(\alpha)\bar{\pi}(\alpha) + F(\alpha)\bar{\pi}^{(1)}(\alpha) + G^{(1)}(\alpha) \\ &= \left(F^{(1)}(\alpha) + (F(\alpha))^2\right)\bar{\pi}(\alpha) + F(\alpha)G(\alpha) + G^{(1)}(\alpha).\end{aligned}$$

This way, we can compute *all* $\bar{\pi}^{(\ell)}(\alpha)$ recursively in terms of $\bar{\pi}(\alpha)$. Concretely (Check!), $\bar{\pi}^{(\ell)}(\alpha) = A_\ell(\alpha)\,\bar{\pi}(\alpha) + B_\ell(\alpha)$, where $A_\ell(\cdot)$ and $B_\ell(\cdot)$ follow by

$$A_{\ell+1}(\alpha) = A'_\ell(\alpha) + A_\ell(\alpha)\,F(\alpha), \quad B_{\ell+1}(\alpha) = A_\ell(\alpha)\,G(\alpha) + B'_\ell(\alpha);$$

recursion is initialized by $A_1(\alpha) = F(\alpha)$ and $B_1(\alpha) = G(\alpha)$.

Relaxation of exponentiality assumptions, ctd.

- When upward jumps are distributed as mixture of exponentials, with density

$$\sum_{i=1}^{k} g_i \, e^{-\mu_i v}$$

  for some $k \in \mathbb{N}$, constants $g_1, \ldots, g_k$, positive parameters $\mu_1, \ldots, \mu_k$ (such that $g_1/\mu_1 + \cdots + g_k/\mu_k$ equals 1), and $v \geqslant 0$: analysis can be extended immediately.

  New functions $F(\cdot)$ and $G(\cdot)$ have poles at $\mu_1, \ldots, \mu_k$; function $G(\cdot)$ contains unknowns $\bar{\pi}(\mu_1), \ldots, \bar{\pi}(\mu_k)$. Resulting $k+1$ unknowns (i.e., $\bar{\pi}(\mu_1), \ldots, \bar{\pi}(\mu_k)$ and $p(0)$) can be determined as before.
- When upward jumps are Erlang distributed: analysis becomes *much* harder; see short account in book.

CHAPTER V: ALTERNATING NET CUMULATIVE CLAIM PROCESS

Alternating net cumulative claim process: main ideas

This chapter: net cumulative claim process displays different behavior above and below threshold $v \in (-\infty, u)$, with $u > 0$ denoting initial reserve level.

Denote resulting net cumulative claim process by $Y_v(t)$ and its running maximum process by $\bar{Y}_v(t)$, and focus on evaluating the ruin probability, i.e.,

$$p(u, v, t) := \mathbb{P}(\bar{Y}_v(t) \geqslant u).$$

Alternating net cumulative claim process: model

Model description:

- When $Y_v(t)$ is below v, claim arrival rate is λ_-, premium rate is r_- and claims have LST $b_-(\alpha)$ (also when claim under consideration is such that corresponding jump process $\bar{Y}_v(t)$ exceeds v).
- When $Y_v(t)$ is above v, claim arrival rate is λ_+, premium rate is r_+ and claims have LST $b_+(\alpha)$.

We focus on the (somewhat more complicated) variant that $v \in (0, u)$; case that $v \in (-\infty, 0]$ can be dealt with analogously.

Object of interest: probability $p(u, v, T_\beta)$ of ruin before exponentially distributed epoch T_β.

Scale functions

Consider net cumulative claim process $Y(t)$ in *non-alternating setting*, i.e., with claim arrival rate λ, premium rate r, and claim-size distribution having LST $b(\alpha)$.

We focus on computing, for $u_- > 0$, $u_+ \geqslant 0$ and $\beta \geqslant 0$,

$$\delta_-(u_-, u_+, \beta) := \mathbb{P}(\sigma(u_-) \leqslant \min\{\tau(u_+), T_\beta\}),$$
$$\delta_+(u_-, u_+, \beta) := \mathbb{P}(\tau(u_+) \leqslant \min\{\sigma(u_-), T_\beta\});$$

here $\tau(u_+)$ is first epoch that $Y(t)$ enters $[u_+, \infty)$ and $\sigma(u_-)$ is first epoch that $Y(t)$ enters $(-\infty, -u_-]$. Note: observe that $Y(\sigma(u_-)) = -u_-$ (Why?).

Laplace exponent $\varphi(\alpha)$ of the process $Y(t)$ is defined as before: $\varphi(\alpha) = r\alpha - \lambda(1 - b(\alpha))$.

Intermediate goal: evaluate $\delta_-(u_-, u_+, \beta)$ and $\delta_+(u_-, u_+, \beta)$.

Scale functions, ctd.

Define *scale function* $W^{(\beta)}(u)$ as the function whose Laplace-Stieltjes transform is

$$\int_0^\infty e^{-\alpha u} W^{(\beta)}(u)\,du = \frac{1}{\varphi(\alpha)-\beta}.$$

(Exists; see Kyprianou book.)

Second scale function:

$$Z^{(\beta)}(u) := 1 + \beta \int_0^u W^{(\beta)}(x)\,dx.$$

Swapping order of integrals:

$$\int_0^\infty e^{-\alpha u} Z^{(\beta)}(u)\,du = \frac{1}{\alpha} + \beta \int_0^\infty \int_x^\infty e^{-\alpha u}\, W^{(\beta)}(x)\,du\,dx$$
$$= \frac{1}{\alpha} + \frac{\beta}{\alpha}\frac{1}{\varphi(\alpha)-\beta}.$$

Scale functions, ctd.

In Chapter 1 we characterized distribution of $\bar{Y}(T_\beta)$ in terms of the Laplace exponent $\varphi(\alpha)$ and its inverse $\psi(\beta)$. First lemma: alternative representation.

Lemma

For any $u > 0$ and $\beta \geqslant 0$,

$$\mathbb{P}(\bar{Y}(T_\beta) > u) = Z^{(\beta)}(u) - \frac{\beta}{\psi(\beta)} W^{(\beta)}(u).$$

Proof. Verify that transform (with respect to u, that is) coincides with $\pi(\alpha, \beta)$. This requires an easy calculation (Check!).

Scale functions, ctd.

Second lemma: useful alternative expressions for the target quantities.

Lemma

For any $u_- > 0$, $u_+ \geqslant 0$, and $\beta \geqslant 0$,

$$\delta_-(u_-, u_+, \beta) = \mathbb{E}\Big(e^{-\beta\sigma(u_-)}\,1\{\sigma(u_-) \leqslant \tau(u_+)\}\Big),$$
$$\delta_+(u_-, u_+, \beta) = \mathbb{E}\Big(e^{-\beta\tau(u_+)}\,1\{\tau(u_+) \leqslant \sigma(u_-)\}\Big).$$

Proof. We establish claim for $\delta_-(u_-, u_+, \beta)$; other claim analogous. Applying integration by parts,

$$\begin{aligned}\delta_-(u_-, u_+, \beta) &= \int_0^\infty \beta e^{-\beta t}\,\mathbb{P}(\sigma(u_-) \leqslant t, \sigma(u_-) \leqslant \tau(u_+))\,dt \\ &= \int_0^\infty e^{-\beta t}\,\mathbb{P}(\sigma(u_-) \in dt, \sigma(u_-) \leqslant \tau(u_+)) \\ &= \mathbb{E}\Big(e^{-\beta\sigma(u_-)}\,1\{\sigma(u_-) \leqslant \tau(u_+)\}\Big).\end{aligned}$$

Scale functions, ctd.

Third lemma: translation in terms of scale functions for infinite horizon case.

Lemma

Assume $\mathbb{E}\, Y(1) < 0$*, or equivalently* $\varphi'(0) > 0$*. Then, for any* $u_- > 0$*,* $u_+ \geqslant 0$*,*

$$\delta_-(u_-, u_+, 0) = \frac{W^{(0)}(u_+)}{W^{(0)}(u_+ + u_-)}.$$

Scale functions, ctd.

Proof. Consider identity

$$\mathbb{P}(\bar{Y}(\infty) < u_+) = \mathbb{P}(\bar{Y}(\infty) < u_+ + u_-)\,\mathbb{P}(\sigma(u_-) \leqslant \tau(u_+)),$$

where we use $Y(\sigma(u_-)) = -u_-$ & strong Markov property. Due to $\mathbb{E}\,Y(1) < 0$ both $\mathbb{P}(\bar{Y}(\infty) < u_+)$ and $\mathbb{P}(\bar{Y}(\infty) < u_+ + u_-)$ are positive so that

$$\delta_-(u_-, u_+, 0) = \frac{\mathbb{P}(\bar{Y}(\infty) < u_+)}{\mathbb{P}(\bar{Y}(\infty) < u_+ + u_-)}.$$

Hence: left to prove that $\mathbb{P}(\bar{Y}(\infty) < u)$ is proportional to $W^{(0)}(u)$.

Scale functions, ctd.

The proportionality we show by establishing that transforms of both objects are proportional. Indeed, by Corollary 1.1,

$$\int_{0+}^{\infty} e^{-\alpha u}\,\mathbb{P}(\bar{Y}(\infty) < u)\,du = \frac{1}{\alpha}\mathbb{P}(\bar{Y}(\infty) = 0) + \frac{1}{\alpha}\int_{0+}^{\infty} e^{-\alpha u}\,\mathbb{P}(\bar{Y}(\infty) \in du)$$

$$= \frac{1}{\alpha}\int_{0}^{\infty} e^{-\alpha u}\,\mathbb{P}(\bar{Y}(\infty) \in du) = \frac{\varphi'(0)}{\varphi(\alpha)},$$

which is proportional to $1/\varphi(\alpha)$, i.e., transform of $W^{(0)}(u)$.

Scale functions, ctd.

Theorem

For any $u_- > 0$, $u_+ \geqslant 0$ and $\beta > 0$,

$$\delta_-(u_-, u_+, \beta) = \frac{W^{(\beta)}(u_+)}{W^{(\beta)}(u_+ + u_-)}.$$

Scale functions, ctd.

Proof. Study process $Y(t)$, for $\beta > 0$, under exponential change-of-measure with 'twist' $-\psi(\beta) < 0$: calling alternative probability model $\mathbb{Q}$, Laplace exponent under $\mathbb{Q}$ is

$$\varphi_{\mathbb{Q}}(\alpha) = \varphi(\alpha + \psi(\beta)) - \varphi(\psi(\beta)) = \varphi(\alpha + \psi(\beta)) - \beta.$$

New process has negative mean: $\varphi'(\psi(0)) > 0$, in combination with (i) the right inverse $\psi(\beta)$ is increasing in β and (ii) $\varphi(\alpha)$ is increasing for $\alpha > \psi(\beta)$, yields

$$\mathbb{E}_{\mathbb{Q}} Y(1) = -\varphi'_{\mathbb{Q}}(0) = -\varphi'(\psi(\beta)) < 0.$$

See Figure.

Scale functions, ctd.

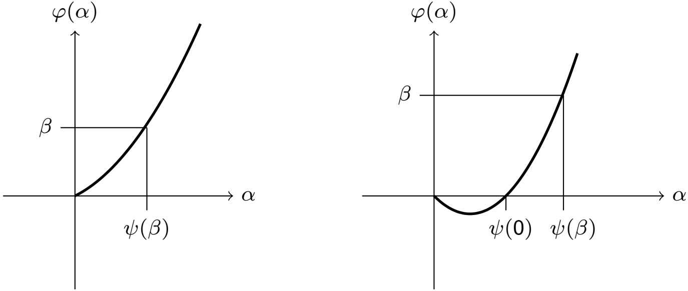

Figure: Functions $\varphi(\alpha)$ with $\varphi'(0) > 0$ (left panel) and $\varphi(\alpha)$ with $\varphi'(0) < 0$ (right panel). In former case $\psi(\beta) > \psi(0) = 0$, whereas in latter case $\psi(\beta) > \psi(0) > 0$. Observe that in both cases $\varphi'(\psi(\beta)) > 0$.

Scale functions, ctd.

Likelihood ratio connecting $\mathbb{P}$ and $\mathbb{Q}$:

$$\frac{d\mathbb{P}(Y(t)=x)}{d\mathbb{Q}(Y(t)=x)} = e^{\beta t} e^{\psi(\beta)\,x};$$

to see this, observe that

$$\int_{-\infty}^{\infty} e^{-\alpha x}\,\mathbb{Q}(Y(t)\in dx) = \mathbb{E}_{\mathbb{Q}}\,e^{-\alpha Y(t)} = \mathbb{E}\,e^{-(\alpha+\psi(\beta))Y(t)-\beta t}$$

$$= e^{-\beta t}\int_{-\infty}^{\infty} e^{-(\alpha+\psi(\beta))x}\,\mathbb{P}(Y(t)\in dx).$$

Third Lemma, which we can apply because $\mathbb{E}_{\mathbb{Q}}Y(1) < 0$:

$$\mathbb{Q}(\sigma(u_-)\leqslant\tau(u_+)) = \frac{\mathbb{Q}(\bar{Y}(\infty)<u_+)}{\mathbb{Q}(\bar{Y}(\infty)<u_++u_-)}.$$

On the other hand, by applying likelihood ratio and Second Lemma,

$$\mathbb{Q}(\sigma(u_-)\leqslant\tau(u_+)) = \mathbb{E}\Big(e^{-\beta\sigma(u_-)}e^{-\psi(\beta)\,Y(\sigma(u_-))}1\{\sigma(u_-)\leqslant\tau(u_+)\}\Big)$$

$$= e^{\psi(\beta)u_-}\,\delta_-(u_-,u_+,\beta).$$

Scale functions, ctd.

Combining above findings,

$$\delta_-(u_-, u_+, \beta) = e^{-\psi(\beta)\, u_-} \frac{\mathbb{Q}(\bar{Y}(\infty) < u_+)}{\mathbb{Q}(\bar{Y}(\infty) < u_+ + u_-)}$$
$$= \frac{e^{\psi(\beta)\, u_+}\, \mathbb{Q}(\bar{Y}(\infty) < u_+)}{e^{\psi(\beta)\, (u_+ + u_-)}\, \mathbb{Q}(\bar{Y}(\infty) < u_+ + u_-)}.$$

Left to prove: $e^{\psi(\beta)\, u}\, \mathbb{Q}(\bar{Y}(\infty) < u)$ is proportional to $W^{(\beta)}(u)$. Idea: show that their transforms match up to multiplicative constant:

$$\int_{0+}^{\infty} e^{-\alpha u} e^{\psi(\beta)\, u}\, \mathbb{Q}(\bar{Y}(\infty) < u)\, du = \frac{\varphi'_{\mathbb{Q}}(0)}{\varphi_{\mathbb{Q}}(\alpha - \psi(\beta))} = \frac{\varphi'(\psi(\beta))}{\varphi(\alpha) - \beta},$$

which is proportional to $1/(\varphi(\alpha) - \beta)$. Stated follows by recalling that $1/(\varphi(\alpha) - \beta)$ is transform of $W^{(\beta)}(u)$.

Scale functions, ctd.

Theorem

For any $u_- > 0$, $u_+ \geqslant 0$ *and* $\beta > 0$,

$$\delta_+(u_-, u_+, \beta) = Z^{(\beta)}(u_+) - Z^{(\beta)}(u_+ + u_-)\frac{W^{(\beta)}(u_+)}{W^{(\beta)}(u_+ + u_-)}.$$

Scale functions, ctd.

Proof. We first decompose

$$\delta_+(u_-,u_+,\beta) = \mathbb{E}\big(e^{-\beta\tau(u_+)}\,1\{\tau(u_+)<\infty\}\big) - \\ \mathbb{E}\big(e^{-\beta\tau(u_+)}\,1\{\sigma(u_-)<\tau(u_+)\}\big).$$

First term, by Equation (1.2) and equivalence of $\{\tau(u)<t\}$ and $\{\bar{Y}(t)>u\}$,

$$\mathbb{E}\big(e^{-\beta\tau(u_+)}\,1\{\tau(u_+)<\infty\}\big) = \mathbb{P}(T_\beta>\tau(u_+)) = \mathbb{P}(\bar{Y}(T_\beta)>u_+),$$

which we can evaluate in terms of scale functions relying on First Lemma. In addition, using the strong Markov property,

$$\begin{aligned}\mathbb{E}\big(e^{-\beta\tau(u_+)}&\,1\{\sigma(u_-)<\tau(u_+)\}\big)\\ &= \mathbb{P}(\sigma(u_-)<\tau(u_+)<T_\beta)\\ &= \mathbb{P}(\sigma(u_-)<T_\beta,\sigma(u_-)<\tau(u_+))\,\mathbb{P}(\tau(u_++u_-)<T_\beta).\end{aligned}$$

Scale functions, ctd.

Combining above findings:

$$\delta_+(u_-, u_+, \beta) = \mathbb{P}(\bar{Y}(T_\beta) > u_+) - \delta_-(u_-, u_+, \beta)\,\mathbb{P}(\bar{Y}(T_\beta) > u_+ + u_-).$$

Using previous Theorem and First Lemma, this yields desired expression. Here use that (Check!)

$$\begin{aligned}
&\mathbb{P}(\bar{Y}(T_\beta) > u_+) - \delta_-(u_-, u_+, \beta)\,\mathbb{P}(\bar{Y}(T_\beta) > u_+ + u_-) \\
&\qquad = Z^{(\beta)}(u_+) - \frac{\beta}{\psi(\beta)} W^{(\beta)}(u_+) - \\
&\qquad\quad \frac{W^{(\beta)}(u_+)}{W^{(\beta)}(u_+ + u_-)} \left(Z^{(\beta)}(u_+ + u_-) - \frac{\beta}{\psi(\beta)} W^{(\beta)}(u_+ + u_-) \right),
\end{aligned}$$

which equals right-hand side of claimed equality.

Decomposition

Goal: evaluating $p(u, v, T_\beta)$, i.e., probability of $Y_v(t)$ exceeding level u before time T_β. We do so working with a *decomposition.*

Key quantity is *first passage time*

$$\tau(w) := \inf\{t \geqslant 0 : Y_v(t) \geqslant w \,|\, Y_v(0) = 0\}.$$

In addition, for $y \in (v, u)$,

$$\tau_y(u) := \inf\{t \geqslant 0 : Y_v(t) \geqslant u \,|\, Y_v(0) = y\},$$
$$\sigma_y(v) := \inf\{t \geqslant 0 : Y_v(t) \leqslant v \,|\, Y_v(0) = y\}.$$

Note that in definition of $\sigma_y(v)$ we could have replaced '$\leqslant v$' by '$= v$' (Why?).

Decomposition, ctd.

Crucial role is played by *overshoot* over level v, jointly with indicator function of the event of $Y_v(t)$ exceeding v before time T_β. Introduce

$$k(v,t,\gamma) := \mathbb{E}\left(e^{-\gamma\,(Y_v(\tau(v))-v)}1\{\tau(v)\leqslant t\}\right).$$

Later: evaluate double transform of $k(v,t,\gamma)$, or, equivalently,

$$\kappa(\alpha,\beta,\gamma) := \int_0^\infty e^{-\alpha v}\,k(v,\,T_\beta,\gamma)\,dv.$$

This, applying Laplace inversion, allows evaluation of

$$\mathbb{P}(Y_v(\tau(v)) - v \in dy, \tau(v)\leqslant T_\beta);$$

in the sequel denote this density by $h(v,y,\beta)$.

Decomposition, ctd.

Definition of $q(u,v,t)$ and $\bar{q}(u,v,t)$, with $v < u$, to be evaluated later.

$\bar{q}(u,v,t)$: probability that starting in v, first a level above v is attained (before t), and then v is reached again (before t), before u is exceeded (also before t). Formally, $\bar{q}(u,v,t) := \mathbb{P}(\mathscr{E}(u,v,t) \,|\, Y_v(0) = v)$, with

$$\mathscr{E}(u,v,t) := \left\{ \begin{array}{l} s_1 := \inf\{s > 0 : Y_v(s) > v\} \leqslant t, \\ s_2 := \inf\{s > s_1 : Y_v(s) = v\} \leqslant t, \\ \forall s \in (s_1, s_2) : Y_v(s) < u \end{array} \right\}.$$

$q(u,v,t)$: probability that starting in v, level u is exceeded (before t), before v is reached from above (also before t). Formally, $q(u,v,t) := \mathbb{P}(\mathscr{F}(u,v,t) \,|\, Y_v(0) = 0)$, with

$$\mathscr{F}(u,v,t) := \left\{ \begin{array}{l} s_1 := \inf\{s > 0 : Y_v(s) > v\} \leqslant t, \\ s_2 := \inf\{s \geqslant s_1 : Y_v(s) \geqslant u\} \leqslant t, \\ \forall s \in [s_1, s_2] : Y_v(s) > v \end{array} \right\};$$

also includes case in which at first time v is exceeded, u is exceeded too.

Decomposition, ctd.

Considering target probability $p(u, v, T_\beta)$, there are three (disjoint) ways to exceed u, starting with $Y_v(0) = 0$.

1. Level v can be exceeded (before T_β) with overshoot that is larger than $u - v$. This leads to contribution

$$p_1(u, v, T_\beta) := \int_{u-v}^{\infty} h(v, y, \beta)\, dy.$$

2. Level v is exceeded with overshoot that lies between 0 and $u - v$, but from that point on u is exceeded before v is reached (and all these events before T_β). This corresponds to contribution

$$p_2(u, v, T_\beta) := \int_0^{u-v} h(v, y, \beta)\, \delta_{+,y}(u, v, \beta)\, dy,$$

with $\delta_{+,y}(u, v, \beta) := \mathbb{P}(\tau_y(u) \leqslant \min\{\sigma_y(v), T_\beta\})$ evaluated later.

Decomposition, ctd.

3. Level v can be exceeded with overshoot that lies between 0 and $u - v$, but from that point on v is reached before u is exceeded (and all these events occur before T_β). From that point on, geometric number of attempts of exceeding u starting at level v; in each of these attempts, the process first has to exceed level v again, and after that u should be exceeded before returning to v (all these events occurring before T_β). This leads to contribution

$$\begin{aligned} p_3(u,v,T_\beta) &:= \int_0^{u-v} h(v,y,\beta)\,\delta_{-,y}(u,v,\beta)\,dy \times \sum_{k=0}^{\infty} q(u,v,T_\beta)\left(\bar{q}(u,v,T_\beta)\right)^k \\ &= \frac{q(u,v,T_\beta)}{1-\bar{q}(u,v,T_\beta)} \int_0^{u-v} h(v,y,\beta)\,\delta_{-,y}(u,v,\beta)\,dy, \end{aligned}$$

with $\delta_{-,y}(u,v,\beta) := \mathbb{P}(\sigma_y(v) \leqslant \min\{\tau_y(u), T_\beta\})$ evaluated later.

Decomposition, ctd.

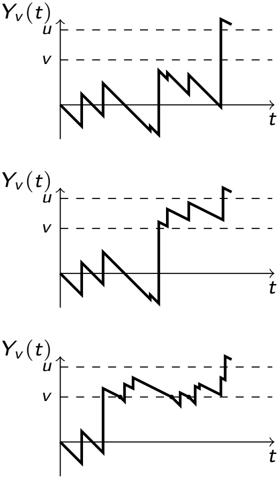

Figure: Process $Y_v(t)$. Top panel: Scenario 1, middle panel: Scenario 2, bottom panel: Scenario 3 (black dots indicating start of new attempt to exceed level u starting at level v).

Decomposition, ctd.

Theorem

For any $u > 0$*,* $v \in (0, u)$*, and* $\beta > 0$*,*

$$p(u, v, T_\beta) = p_1(u, v, T_\beta) + p_2(u, v, T_\beta) + p_3(u, v, T_\beta).$$

Computation of auxiliary objects

We conclude by evaluating all objects needed in decomposition of Theorem:

- density $h(y, v, \beta)$ (through the associated transform $\kappa(\alpha, \beta, \gamma)$),
- probabilities $\delta_{-,y}(u, v, \beta)$ and $\delta_{+,y}(u, v, \beta)$,
- probabilities $q(u, v, T_\beta)$ and $\bar{q}(u, v, T_\beta)$.

Computation of auxiliary objects, ctd.

Evaluation of $\kappa(\alpha,\beta,\gamma)$: as in Exercise 1.2.

With $\varphi_-(\alpha) := r_-\alpha - \lambda_-(1 - b_-(\alpha))$, and $\psi_-(\beta)$ right inverse of $\varphi_-(\alpha)$:

$$\kappa(\alpha,\beta,\gamma) = \frac{\lambda_-}{\varphi_-(\alpha)-\beta}\left(\frac{b_-(\psi_-(\beta)) - b_-(\gamma)}{\gamma - \psi_-(\beta)} - \frac{b_-(\alpha) - b_-(\gamma)}{\gamma - \alpha}\right).$$

Computation of auxiliary objects, ctd.

Evaluation of $\delta_{-,y}(u,v,\beta)$ and $\delta_{+,y}(u,v,\beta)$: use scale functions.

Using results that we derived,

$$\delta_{-,y}(u,v,\beta) = \frac{W_+^{(\beta)}(u-y)}{W_+^{(\beta)}(u-v)},$$

with $W_+^{(\beta)}(u)$ such that, with $\varphi_+(\alpha) := r_+\alpha - \lambda_+(1-b_+(\alpha))$,

$$\int_0^\infty e^{-\alpha u} W_+^{(\beta)}(u)\,du = \frac{1}{\varphi_+(\alpha)-\beta}.$$

Also,

$$\delta_{+,y}(u,v,\beta) = Z_+^{(\beta)}(u-y) - Z_+^{(\beta)}(u-v)\frac{W_+^{(\beta)}(u-y)}{W_+^{(\beta)}(u-v)}.$$

Computation of auxiliary objects, ctd.

Evaluation of $q(u,v,T_\beta)$ and $\bar{q}(u,v,T_\beta)$: with $\delta_{-,y}(u,v,\beta)$ and $\delta_{+,y}(u,v,\beta)$ as given above, it is seen that (Check!)

$$q(u,v,T_\beta) = \int_0^{u-v} h(0+,y,T_\beta)\,\delta_{+,y}(u,v,\beta)\,dy + \int_{u-v}^{\infty} h(0+,y,T_\beta)\,dy$$

and

$$\bar{q}(u,v,T_\beta) = \int_0^{u-v} h(0+,y,T_\beta)\,\delta_{-,y}(u,v,\beta)\,dy.$$

Density $h(0+,y,T_\beta)$ can be determined as pointed out earlier.

CHAPTER VI: LEVEL-DEPENDENT DYNAMICS

Level-dependent dynamics: main ideas

This chapter: behavior of net cumulative claim process depends on current reserve level 'in a continuous manner'.

We consider CL model, but now with claim arrival rate and premium rate equal to $\lambda(x)$ and $r(x)$, respectively, when the surplus level is x.

Assume: $r(0) = 0$ and $r(x) > 0$ for all $x > 0$.

Level-dependent dynamics: main ideas, ctd.

Reserve level process obeys integral equation:

$$X_u(t) = u + \int_0^t r(X_u(s))\, ds - \sum_{i=1}^{N(t)} B_i,$$

where claim arrival process $N(t)$ is such that Poisson arrival rate at time t is $\lambda(X_u(t))$.

More precisely, as $\Delta t \downarrow 0$,

$$\mathbb{P}(N(t+\Delta t) - N(t) = 1 \,|\, X_u(s), s \in [0,t]) = \lambda(X_u(t))\,\Delta t + o(\Delta t),$$

and

$$\mathbb{P}(N(t+\Delta t) - N(t) = 0 \,|\, X_u(s), s \in [0,t]) = 1 - \lambda(X_u(t))\,\Delta t + o(\Delta t),$$

where probability of two or more arrivals in interval of length Δt is $o(\Delta t)$.

Level-dependent dynamics: main ideas, ctd.

Objective: analysis of all-time ruin probability $p(u)$, i.e., probability of $X_u(t)$ ever dropping below 0.

For general functions $\lambda(x)$ and $r(x)$ evaluation of time-dependent ruin probability $p(u, t)$ is beyond reach, except in special cases.

Level-dependent premium rate: duality

First assume $\lambda(x) \equiv \lambda$.

Construct dual queueing process $Q(s)$, for $s \in [0, t]$, as follows:

- Apply time reversal on the interval $[0, t]$. This concretely means that the process' jumps are now *positive*.
- Apply reflection at zero to prevent the process from attaining negative values.
- Start the queue with a zero workload: $Q(0) = 0$.

Workload dynamics are governed by

$$Q(t) = \sum_{i=1}^{N(t)} B_i - \int_0^t r(Q(s))\, ds.$$

Level-dependent premium rate: duality

Claim: finite-time ruin probability $p(u, t)$ equals probability of workload level $Q(t)$ exceeding u (where $Q(0) = 0$); analogously, all-time ruin probability $p(u)$ equals probability of stationary workload level $Q(\infty)$ exceeding u.

Let $\tau(u)$ denote first time that reserve level $X_u(t)$ attains a non-positive value, i.e., the *ruin time*.

Theorem

For any $t > 0$, the events $\{\tau(u) \leqslant t\}$ and $\{Q(t) > u\}$ coincide. In particular, the events $\{\tau(u) < \infty\}$ and $\{Q(\infty) > u\}$ coincide.

Level-dependent premium rate: duality, ctd.

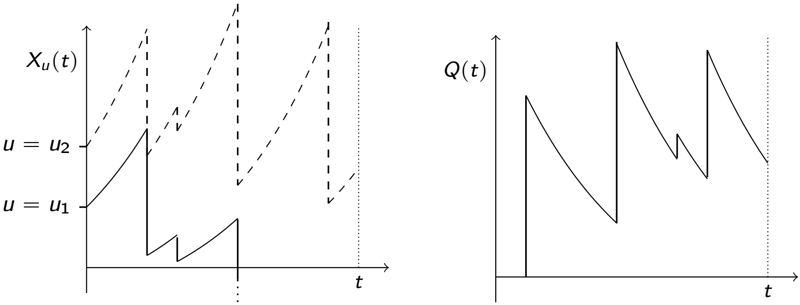

Figure: Left panel: reserve level process $X_u(t)$ for initial surplus u_1 (solid lines) and for initial level u_2 (dashed lines). Right panel: constructed workload process $Q(t)$, with time-reversed arrival process.

Level-dependent premium rate: duality, ctd.

Proof. Relies on a *sample-path comparison technique.*

Let there be N claims in the reserve level process $X_u(t)$ between 0 and t (which is Poisson distributed with parameter λt); call these times t_1 up to t_N. Because of time reversal, jumps in dual queueing process $Q(t)$ happen at times $t_n^\star := t - t_{N-n+1}$, for $n = 1, \dots, N$.

Claims $B_1, \dots, B_N$ in reserve level process $X_u(t)$ correspond to upward jumps in the queueing process $Q(t)$ of size $B_n^\star = B_{N-n+1}$.

Let deterministic function $x_u(s)$ solve $x_u'(s) = r(x_u(s))$ under $x_u(0) = u$. Evidently, there is monotonicity as function of initial surplus level: if $u_1 < u_2$, then $x_{u_1}(s) < x_{u_2}(s)$.

Proof of equivalence of $\{\tau(u) \leqslant t\}$ and $\{Q(t) > u\}$: two inclusions.

Level-dependent premium rate: duality, ctd.

- First consider scenario that $Q(t) > u$, corresponding to path of $X_{u_1}(t)$ (i.e., solid graph in left panel). Due to monotonicity,

$$Q(t_N^\star-) = x_{Q(t)}(t_1) - B_1 > x_u(t_1) - B_1 = X_u(t_1).$$

If $Q(t_N^\star-) = 0$, then $X_u(t_1) < 0$, so that indeed $\tau(u) \leqslant t$. If $Q(t_N^\star-) > 0$, iterate above argument to conclude that $Q(t_{N-1}^\star) > X_u(t_2)$:

$$\begin{aligned} Q(t_{N-1}^\star-) &= x_{Q(t_N^\star-)}(t_2 - t_1) - B_2 \\ &> x_{X_u(t_1)}(t_2 - t_1) - B_2 = X_u(t_2). \end{aligned}$$

Again distinguish $Q(t_{N-1}^\star-) = 0$ and $Q(t_{N-1}^\star-) > 0$. Former case: $X_u(t_2) < 0$ and hence $\tau(u) \leqslant t$.
Continuing along these lines, due to $Q(t_1^\star-) = 0$, this procedure will eventually yield $t_j^\star$ such that $Q(t_j^\star-) = 0$. Hence, for this j we have that $X_u(t_{N-j+1}) < 0$, so that $\tau(u) \leqslant t$, as desired.

Level-dependent premium rate: duality, ctd.

- Conversely, now suppose that $Q(t) \leqslant u$, corresponding to path of $X_{u_2}(t)$ (i.e., dashed graph in left panel). Then, using monotonicity once more,

$$Q(t_N^\star -) = x_{Q(t)}(t_1) - B_1 \leqslant x_u(t_1) - B_1 = X_u(t_1).$$

  This relation can be iterated in same way as before, to obtain $Q(t_j^\star -) \leqslant X_u(t_{N-j+1})$, for all $j \in \{1, \ldots, N\}$.

  Together with $Q(s) \geqslant 0$, this implies that at all claim arrivals reserve level process is non-negative. As ruin can only occur at claim arrivals, this means that no ruin occurs in $[0, t)$, i.e., that $\tau(u) > t$.

Level-dependent premium rate: distribution

Justified by duality, describe distribution of stationary workload $Q(\infty)$. $f(y)$: density of stationary workload. Observe: equals $-p'(y)$ by virtue of duality. $F(0)$: probability that stationary workload is 0.

Theorem

For $y > 0$,

$$r(y)f(y) = \lambda \int_{0+}^{y} \mathbb{P}(B > y - z) f(z)\, dz + \lambda\, F(0)\, \mathbb{P}(B > y).$$

Proof. Left-hand side can be interpreted as probability flux through the level y from above, and right-hand side as probability flux through y from below.

Level-dependent premium rate: distribution, ctd.

Next challenge: compute density $f(y)$ from integral equation (*Volterra integral equation of second kind*). We restrict ourselves to case $F(0) > 0$.

Introduce $g(y) := \lambda \mathbb{P}(B > y)$ for $y \geqslant 0$ and *kernel* $K(y, z) := g(y - z)/r(y)$ for $0 \leqslant z < y < \infty$. We obtain alternative representation

$$f(y) = K(y, 0)F(0) + \int_{0+}^{y} K(y, z)f(z)\, dz.$$

Define the kernels $K_n(x, y)$ iteratively by $K_1(x, y) := K(x, y)$ and

$$K_n(x, y) := \int_{y}^{x} K_{n-1}(x, z)\, K(z, y)\, dz$$

for $0 \leqslant y < x < \infty$ and $n \in \{2, 3, \ldots\}$.

Level-dependent premium rate: distribution, ctd.

Solve iteratively:

$$f(y) = K(y,0)F(0) + \int_{0+}^{y} K(y,z)\left(K(z,0)F(0) + \int_{0+}^{z} K(z,w)f(w)\,dw\right)dz$$
$$= \cdots = F(0)\sum_{n=1}^{\infty} K_n(y,0).$$

Level-dependent premium rate: distribution, ctd.

Convergence of sum follows from following Lemma, implying that $K^\star(x,y) := \sum_{n=1}^\infty K_n(x,y)$ is well-defined.

First introduce, for $0 \leqslant y < x < \infty$,

$$R(x,y) := \int_y^x \frac{1}{r(w)}\,dw.$$

Represents time to go from level x to level $y < x$ in absence of arrivals.

Lemma

For $0 \leqslant y < x < \infty$ and $n \in \{1, 2, \ldots\}$,

$$K_n(x,y) \leqslant \frac{\lambda^n R(x,y)^{n-1}}{r(x)(n-1)!}.$$

Level-dependent premium rate: distribution, ctd.

Proof. By induction. For $n = 1$ stated follows from $g(x-y) \leqslant \lambda$: we thus have that $K(x,y) \leqslant \lambda / r(x)$.
Now suppose claim holds for $n-1$. Then, using induction hypothesis,

$$K_n(x,y) = \int_y^x K_{n-1}(x,z)\, K(z,y)\, dz \leqslant \int_y^x \frac{\lambda^{n-1} R(x,z)^{n-2}}{r(x)(n-2)!}\, \frac{\lambda}{r(z)}\, dz.$$

Observing that

$$\frac{d}{dz} R(x,z) = -\frac{1}{r(z)},$$

we have that RHS equals

$$\left[-\frac{\lambda^n R(x,z)^{n-1}}{r(x)(n-1)!} \right]_{z=y}^{x} = \frac{\lambda^n R(x,y)^{n-1}}{r(x)(n-1)!},$$

as desired.

Level-dependent premium rate: distribution, ctd.

We find stationary workload density (equals minus derivative of ruin probability in associated ruin model, as pointed out earlier).

It uses

$$\xi := 1 + \int_{0+}^{\infty} K^{\star}(y, 0)\, dy.$$

Theorem

If $\xi < \infty$, then $F(0) = 1/\xi$ and, for $y > 0$,

$$f(y) = \frac{K^{\star}(y, 0)}{\xi}.$$

Case of exponentially distributed claims can be done explicitly; see last part of Section 6.2.

Level-dependent premium rate and claim arrival rate

Now: premium rate *and* claim arrival rate are level-dependent. Goal: integro-differential equation for *survival probability* $\bar{p}(u) = 1 - p(u)$.

In this context duality does *not* apply (see Remark 6.2). Therefore: Kolmogorov forward equation method, i.e., Method 4 of Section 1.6.

Looking ahead an infinitesimal amount of time Δt,

$$\bar{p}(u) = (1-\lambda(u)\Delta t)\,\bar{p}(u+r(u)\Delta t)+\lambda(u)\Delta t\int_0^{u-}\bar{p}(u-z)\,\mathbb{P}(B\in dz)+o(\Delta t).$$

Bring $\bar{p}(u + r(u)\Delta t)$ to LHS and divide by Δt. After $\Delta t \downarrow 0$,

$$\begin{aligned} r(u)\bar{p}'(u) &= \lambda(u)\,\bar{p}(u) - \lambda(u)\int_0^{u-}\bar{p}(u-z)\,\mathbb{P}(B\in dz) \\ &= \lambda(u)\,\bar{p}(u) + \lambda(u)\int_0^{u-}\bar{p}(u-z)\,d\mathbb{P}(B>z). \end{aligned}$$

Then apply integration by parts.

Level-dependent premium rate: distribution, ctd.

Write $f(u) := \bar{p}'(u)$.

Theorem

For $u > 0$,

$$r(u)\, f(u) = \lambda(u) \int_{0+}^{u} \mathbb{P}(B > u - z)\, f(z)\, dz + \lambda(u)\, \bar{p}(0)\, \mathbb{P}(B > u).$$

Level-dependent premium rate: distribution, ctd.

Introduce $\zeta(u) := \lambda r(u)/\lambda(u)$.

Equality in Theorem becomes

$$\zeta(u)\, f(u) = \lambda \int_0^u \mathbb{P}(B > u - z)\, f(z)\, dz + \lambda\, \bar{p}(0)\, \mathbb{P}(B > u).$$

Has exact same structure as equality for $\lambda(x) \equiv \lambda$. Hence can again be solved by same type of iteration.

Specific level-dependent model

Variant of CL model, where high surplus leads to increase of claim arrival rate.

Model:

- Let $A_1, A_2, \dots$ be a sequence of i.i.d. $\exp(\lambda)$ rv s.
- When surplus level right after i-th claim arrival is y, then next inter-claim time equals $\max\{0, A_i - cy\}$, where c is positive constant.

Mechanism is such that when surplus level is large, there is a cascade of claims, so that reserve level is pulled down, whereas if surplus level is small, the model effectively behaves as conventional CL model.

Suggests that $p(u) = 1$, as surplus process cannot drift to ∞.

Specific level-dependent model

As before: objective is to evaluate ruin probability over exponentially distributed horizon, i.e., $p(u, T_\beta)$, through its Laplace transform:

$$\pi(\alpha, \beta) := \int_0^\infty e^{-\alpha u} p(u, T_\beta)\, du.$$

Main idea: by conditioning on first claim arrival, we can express $\pi(\alpha, \beta)$ in itself, but evaluated in different arguments.

Two scenarios are relevant:

- If exponentially distributed random variable with parameter λ, say A, is smaller than cu, then next claim arrives instantly. This could lead to instantaneous ruin if its size is larger than u, and alternatively can bring the surplus process down to level between 0 and u.
- A can be larger than cu. Then claim arrives after $A - cu$ time units. Again, this can lead to either immediate ruin, or to surplus level between 0 and u (if time horizon T_β has not been exceeded).

Specific level-dependent model, ctd.

Reasoning of the preceding slide entails

$$p(u, T_\beta) = p_1(u, T_\beta) + p_2(u, T_\beta).$$

Here $p_1(u, T_\beta)$ corresponds to first scenario, i.e.,

$$p_1(u, T_\beta) = (1 - e^{-\lambda cu})\Big(\int_0^u p(u - v, T_\beta)\,\mathbb{P}(B \in dv) + \int_u^\infty \mathbb{P}(B \in dv)\Big),$$

and $p_2(u, T_\beta)$ to second scenario, i.e.,

$$p_2(u, T_\beta) = \int_{cu}^\infty \lambda e^{-\lambda s}\,\mathbb{P}(T_\beta \geqslant s - cu)$$

$$\Big(\int_0^{u+r(s-cu)} p(u + r(s - cu) - v, T_\beta)\,\mathbb{P}(B \in dv) + \int_{u+r(s-cu)}^\infty \mathbb{P}(B \in dv)\Big)\,ds.$$

Specific level-dependent model, ctd.

$\pi_i(\alpha,\beta)$: Laplace transform of $p_i(\cdot\,, T_\beta)$, for $i = 1, 2$.

Focusing on $\pi_1(\alpha,\beta)$, the integral

$$\int_0^\infty e^{-\alpha u}(1-e^{-\lambda c u})\int_0^u p(u-v,\,T_\beta)\,\mathbb{P}(B\in dv)\,du,$$

after swapping the order of integrals and recognizing Laplace transform of a convolution, equals

$$b(\alpha)\pi(\alpha,\beta)-b(\alpha+\lambda c)\pi(\alpha+\lambda c,\beta).$$

Along similar lines,

$$\int_0^\infty e^{-\alpha u}(1-e^{-\lambda c u})\int_u^\infty \mathbb{P}(B\in dv)\,du=\frac{1-b(\alpha)}{\alpha}-\frac{1-b(\alpha+\lambda c)}{\alpha+\lambda c}.$$

Conclude: $\pi_1(\alpha,\beta)$ is sum of these expressions.

Specific level-dependent model, ctd.

Now consider evaluation of $\pi_2(\alpha, \beta)$. We are to calculate two triple integrals, using standard techniques.

First integral equals:

$$\frac{\lambda}{r}\left(\frac{b((\lambda+\beta)/r)\,\pi((\lambda+\beta)/r,\beta)-b(\alpha+\lambda c)\,\pi(\alpha+\lambda c,\beta)}{\alpha+\lambda c-(\lambda+\beta)/r}\right).$$

Second integral equals

$$\frac{\lambda}{\lambda+\beta}\left(\frac{1-b(\alpha+\lambda c)}{\alpha+\lambda c}-\frac{b((\lambda+\beta)/r)-b(\alpha+\lambda c)}{\alpha+\lambda c-(\lambda+\beta)/r}\right).$$

Conclude: $\pi_2(\alpha, \beta)$ is sum of these expressions.

Specific level-dependent model, ctd.

We found, for easily determined functions $F(\alpha,\beta)$, $G(\alpha,\beta)$ and $H(\alpha,\beta)$, a relation of the form

$$\pi(\alpha,\beta) = F(\alpha,\beta)\,\pi(\alpha+\lambda c,\beta) + G(\alpha,\beta) + H(\alpha,\beta)\,\pi((\lambda+\beta)/r,\beta).$$

One can subsequently express $\pi(\alpha+\lambda c,\beta)$ in terms of $\pi(\alpha+2\lambda c,\beta)$, etc. Repeatedly iterating this relation, we obtain an expression for $\pi(\alpha,\beta)$.
In this expression $\kappa(r) := \pi((\lambda+\beta)/r,\beta)$ (with β kept fixed) appears. Expression for $\kappa(r)$ is derived by inserting $\alpha = \alpha(r) := (\lambda+\beta)/r$, and solving the resulting linear equation in $\kappa(r)$.

Specific level-dependent model, ctd.

After some algebra (Exercise 6.4), we find following result. We denote $\alpha_j := \alpha + j\lambda c$ and $\alpha_j(r) := \alpha(r) + j\lambda c$.

Theorem

For any $\alpha \geqslant 0$ and $\beta > 0$,

$$\begin{aligned}\pi(\alpha,\beta) &= G(\alpha,\beta) + H(\alpha,\beta)\,\kappa(r) \\ &\quad + \sum_{j=1}^{\infty}\big(G(\alpha_j,\beta) + H(\alpha_j,\beta)\,\kappa(r)\big)\prod_{i=0}^{j-1}F(\alpha_i,\beta),\end{aligned}$$

where, defining the empty product as 1,

$$\kappa(r) = \frac{\sum_{j=0}^{\infty} G(\alpha_j(r),\beta)\prod_{i=0}^{j-1}F(\alpha_i(r),\beta)}{1-\sum_{j=0}^{\infty} H(\alpha_j(r),\beta)\prod_{i=0}^{j-1}F(\alpha_i(r),\beta)}.$$

CHAPTER VII: MULTIVARIATE RUIN

Multivariate ruin: main ideas

Most of existing ruin theory: primary focus is on *univariate* setting featuring single reserve process. In practice, however, position of insurance firm is often described by multiple, typically correlated, reserve processes.

Multivariate ruin is *hard* — can be dealt with explicitly only under additional assumptions.

Concretely, *ordering* between individual net cumulative claim processes, say $\boldsymbol{Y}(t) \equiv (Y_1(t), \dots, Y_d(t))$ for some $d \in \mathbb{N}$, needs to be imposed.

This chapter: analysis of multivariate ruin under ordering condition. In addition, we derive so-called multivariate Gerber-Shiu metrics (including ruin times, undershoots, and overshoots).

Bivariate case: model

Consider two net cumulative claim processes, say $Y_1(t)$ and $Y_2(t)$, in which claims arrive simultaneously, according to Poisson process with rate λ.

These claims $\boldsymbol{B}_1, \boldsymbol{B}_2, \dots$ are 2-dimensional, componentwise non-negative i.i.d. random vectors, distributed as generic random vector $\boldsymbol{B}$. Their entries are *ordered*:

$$\mathbb{P}\big(B^{(1)} \geqslant B^{(2)}\big) = 1,$$

where $B^{(i)}$ is generic claim size corresponding to $Y_i(t)$.

The premium rate is r for both individual net cumulative claim processes.

Bivariate Laplace exponent is therefore given by

$$\varphi(\boldsymbol{\alpha}) := \log \mathbb{E}\, e^{-\boldsymbol{\alpha}^\top \boldsymbol{Y}(1)} = r\, 1^\top \boldsymbol{\alpha} - \lambda(1 - b(\boldsymbol{\alpha})),$$

with $b(\boldsymbol{\alpha})$ bivariate LST corresponding to random vector $\boldsymbol{B}$.

Bivariate case: model, ctd.

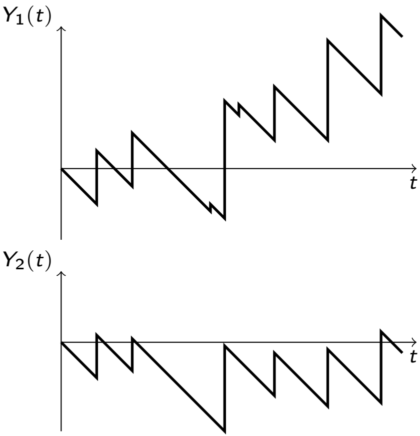

Figure: Net cumulative claim processes $Y_1(t)$ and $Y_2(t)$. Observe that processes are ordered; all jumps in $Y_1(t)$ correspond to simultaneous jumps of at most that size (possibly zero) in $Y_2(t)$.

Bivariate case: model, ctd.

We assumed per-component claim size distributions to be ordered almost surely, whereas premium rates of components are assumed to coincide. We can, however, generalize this (Remark 7.1).

As it turns out, we can work with distinct premium rates r_1 and r_2, but then we have to impose

$$\mathbb{P}\big(B^{(1)}/r_1 \geqslant B^{(2)}/r_2\big) = 1.$$

(Check!)

Bivariate case: key objects

Approach relies on Method 4, discussed in Section 1.6:
we set up Kolmogorov forward equations for bivariate queueing process $\boldsymbol{Q}(t)$ (with $\boldsymbol{Q}(0) = 0$) that is dual of $\boldsymbol{Y}(t)$.

Define

$$\tau_i(u) := \inf\{t \geqslant 0 : Y_i(t) \geqslant u\},$$

for $i = 1, 2$.

Bivariate case: key objects, ctd.

Following lemma shows that ruin in the bivariate risk model (with initial capitals u_1 and u_2) can be expressed in terms of exceedance probabilities (over levels u_1 and u_2) in bivariate dual queueing model.
It justifies that in the sequel we focus on queueing model only.

Lemma

For any $t > 0$,

- *the events $\{\tau_1(u_1) \leqslant t, \tau_2(u_2) \leqslant t\}$ and $\{Q_1(t) > u_1, Q_2(t) > u_2\}$ coincide.*
- *the events $\{\tau_1(u_1) > t, \tau_2(u_2) > t\}$ and $\{Q_1(t) \leqslant u_1, Q_2(t) \leqslant u_2\}$ coincide.*
- *the events $\{\tau_1(u_1) \leqslant t, \tau_2(u_2) > t\}$ and $\{Q_1(t) > u_1, Q_2(t) \leqslant u_2\}$ coincide.*
- *the events $\{\tau_1(u_1) > t, \tau_2(u_2) \leqslant t\}$ and $\{Q_1(t) \leqslant u_1, Q_2(t) > u_2\}$ coincide.*

Bivariate case: key objects, ctd.

Proof. First observe that, based on Theorem 6.1, events $\{\tau_i(u) \leqslant t\}$ and $\{Q_i(t) > u\}$ coincide, for $i = 1, 2$. This directly implies first and second claim. Third claim follows from

$$\{\tau_1(u_1) \leqslant t, \tau_2(u_2) > t\} = \{\tau_1(u_1) \leqslant t\} \backslash \{\tau_1(u_1) \leqslant t, \tau_2(u_2) \leqslant t\},$$

in combination with first claim and fact that events $\{\tau_1(u) \leqslant t\}$ and $\{Q_1(t) > u\}$ coincide. Fourth claim follows by symmetry.

Bivariate case: key objects, ctd.

Our objective is to characterize

$$\kappa_t(\boldsymbol{\alpha}) := \mathbb{E}\, e^{-\boldsymbol{\alpha}^\top \boldsymbol{Q}(t)}.$$

We settle for this object evaluated at exponentially distributed time T_β, for some killing rate β.

Observe: both individual queues are of M/G/1 type, and can therefore be analyzed relying on techniques explained in Chapter 1, but challenge lies in revealing *joint* workload distribution.

Bivariate case: key objects, ctd.

Queueing dynamics in interior of positive quadrant differ from those at boundaries. We therefore also introduce

$$\bar{\kappa}_t(\boldsymbol{\alpha}) := \mathbb{E}\, e^{-\boldsymbol{\alpha}^\top \boldsymbol{Q}(t)} 1\{\boldsymbol{Q}(t) > 0\};$$

the (strict) inequality $\boldsymbol{Q}(t) > 0$ is to be understood componentwise.

Immediate: $Q_1(t) \geqslant Q_2(t)$ almost surely. Hence,

$$\begin{aligned}\bar{\kappa}_t^{(1)}(\alpha_1) &:= \mathbb{E}\, e^{-\boldsymbol{\alpha}^\top \boldsymbol{Q}(t)} 1\{Q_1(t) > 0, Q_2(t) = 0\} \\ &= \mathbb{E}\, e^{-\alpha_1 Q_1(t)} 1\{Q_1(t) > 0, Q_2(t) = 0\}\end{aligned}$$

and

$$\begin{aligned}q_t &:= \mathbb{E}\, e^{-\boldsymbol{\alpha}^\top \boldsymbol{Q}(t)} 1\{Q_1(t) = Q_2(t) = 0\} \\ &= \mathbb{P}(Q_1(t) = Q_2(t) = 0) = \mathbb{P}(Q_1(t) = 0).\end{aligned}$$

Section 1.6: with $\psi_1(\beta)$ right-inverse of $\varphi(\alpha_1, 0)$, for $\beta > 0$,

$$q_{T_\beta} = \frac{\beta}{r\psi_1(\beta)}.$$

Bivariate case: key objects, ctd.

Above transforms translate into transforms related to ruin probabilities, as follows. Define bivariate time-dependent ruin probability:

$$p(\boldsymbol{u}, t) := \mathbb{P}(\tau_1(u_1) \leqslant t, \tau_2(u_2) \leqslant t),$$

and $p_i(u,t)$ is time-dependent ruin probability of firm i, for $i = 1, 2$. Also

$$\pi(\boldsymbol{\alpha}, \beta) := \int_0^\infty \int_0^\infty e^{-\boldsymbol{\alpha}^\top \boldsymbol{u}} p(\boldsymbol{u}, T_\beta)\, du_1\, du_2,$$
$$\pi_i(\alpha, \beta) := \int_0^\infty e^{-\alpha u} p_i(u, T_\beta)\, du.$$

As in Remark 1.2,

$$\kappa_{T_\beta}(\boldsymbol{\alpha}) = 1 - \alpha_1\, \pi_1(\alpha_1, \beta) - \alpha_2\, \pi_2(\alpha_2, \beta) + \alpha_1 \alpha_2\, \pi(\boldsymbol{\alpha}, \beta).$$

Chapter 1: expressions for $\pi_i(\alpha, \beta)$ for $i = 1, 2$. Hence: it suffices to find $\kappa_{T_\beta}(\boldsymbol{\alpha})$ to also find $\pi(\alpha, \beta)$.

Bivariate case: Kolomogorov equations

As in Section 1.6, up to $o(\Delta t)$-terms,

$$\begin{aligned}\bar{\kappa}_{t+\Delta t}(\boldsymbol{\alpha}) + \bar{\kappa}^{(1)}_{t+\Delta t}(\alpha_1) + q_{t+\Delta t} &= \kappa_{t+\Delta t}(\alpha) \\ &= \bar{\kappa}_t(\boldsymbol{\alpha})\big(1 - \lambda\Delta t + \lambda\Delta t\, b(\boldsymbol{\alpha}) + r\,1^\top\boldsymbol{\alpha}\Delta t\big) + \\ &\quad \bar{\kappa}^{(1)}_t(\alpha_1)\big(1 - \lambda\Delta t + \lambda\Delta t\, b(\boldsymbol{\alpha}) + r\alpha_1\Delta t\big) + \\ &\quad q_t\,\big(1 - \lambda\Delta t + \lambda\Delta t\, b(\boldsymbol{\alpha})\big).\end{aligned}$$

Recalling definition of $\varphi(\boldsymbol{\alpha})$, we obtain following differential equation:

Lemma

For any $\boldsymbol{\alpha} \geqslant 0$ *and* $t > 0$,

$$\frac{\partial}{\partial t}\bar{\kappa}_t(\boldsymbol{\alpha}) + \frac{\partial}{\partial t}\bar{\kappa}^{(1)}_t(\alpha_1) + \frac{\partial}{\partial t}q_t$$
$$= \varphi(\boldsymbol{\alpha})\,\bar{\kappa}_t(\boldsymbol{\alpha}) + \big(\varphi(\boldsymbol{\alpha}) - r\alpha_2\big)\,\bar{\kappa}^{(1)}_t(\alpha_1) + \big(\varphi(\boldsymbol{\alpha}) - r1^\top\boldsymbol{\alpha}\big)\,q_t.$$

Bivariate case: derivation of transform.

Next step: transforms at exponentially distributed time T_β.
Multiply full differential equation by density $\beta e^{-\beta t}$, and integrate over $t \geqslant 0$. By Equation (1.7), and realizing that

$$\bar{\kappa}_0(\boldsymbol{\alpha}) = \bar{\kappa}_0^{(1)}(\alpha_1) = 0,$$

we find

$$\beta\Big(\bar{\kappa}_{T_\beta}(\boldsymbol{\alpha}) + \bar{\kappa}_{T_\beta}^{(1)}(\alpha_1) + q_{T_\beta} - 1\Big)$$
$$= \varphi(\boldsymbol{\alpha})\,\bar{\kappa}_{T_\beta}(\boldsymbol{\alpha}) + \big(\varphi(\boldsymbol{\alpha}) - r\alpha_2\big)\,\bar{\kappa}_{T_\beta}^{(1)}(\alpha_1) + \big(\varphi(\boldsymbol{\alpha}) - r1^\top\boldsymbol{\alpha}\big)\,q_{T_\beta}.$$

Bivariate case: derivation of transform, ctd.

After rearranging:

$$\bar{\kappa}_{T_\beta}(\boldsymbol{\alpha}) = -\frac{\big(\varphi(\boldsymbol{\alpha}) - r\alpha_2 - \beta\big)\,\bar{\kappa}^{(1)}_{T_\beta}(\alpha_1) + \big(\varphi(\boldsymbol{\alpha}) - r1^\top\boldsymbol{\alpha} - \beta\big)\,q_{T_\beta} + \beta}{\varphi(\boldsymbol{\alpha}) - \beta},$$

so that we end up with

$$\kappa_{T_\beta}(\boldsymbol{\alpha}) = \frac{r\alpha_2\,\bar{\kappa}^{(1)}_{T_\beta}(\alpha_1) + r1^\top\boldsymbol{\alpha}\,q_{T_\beta} - \beta}{\varphi(\boldsymbol{\alpha}) - \beta}.$$

We lack, however, expression for

$$\bar{\kappa}^{(1)}_{T_\beta}(\alpha_1) = \mathbb{E}\,e^{-\alpha_1 Q_1(T_\beta)}1\{Q_1(T_\beta) > 0, Q_2(T_\beta) = 0\}.$$

Bivariate case: derivation of transform, ctd.

Strategy: any zero of denominator (with positive real part) is necessarily also a zero of the numerator. Rewrite $\varphi(\boldsymbol{\alpha}) - \beta = 0$ as

$$\lambda b(\boldsymbol{\alpha}) = c(\boldsymbol{\alpha}) := \lambda - r1^\top \boldsymbol{\alpha} + \beta.$$

Fixing α_1 with $Re\ \alpha_1 > 0$ and β, due to lemma below we can identify unique $\alpha_2 = \omega_2(\alpha_1, \beta)$ such that $\varphi(\boldsymbol{\alpha}) - \beta = 0$ while $\kappa_{T_\beta}(\boldsymbol{\alpha})$ should be finite. Proof relies on Rouché's theorem.

Lemma

For every α_1 with $Re\,\alpha_1 > 0$ and $\beta > 0$, there exists a unique $\alpha_2 = \omega_2(\alpha_1, \beta)$ with $Re\,\omega_2(\alpha_1, \beta) > Re\,(-\alpha_1)$ that satisfies $\lambda b(\boldsymbol{\alpha}) = c(\boldsymbol{\alpha})$. For any $\beta > 0$, the function $\alpha_1 \mapsto \omega_2(\alpha_1, \beta)$ is analytic in $Re\,\alpha_1 > 0$.

Bivariate case: derivation of transform, ctd.

By Lemma, we obtain by equating numerator to 0:

$$r\omega_2(\alpha_1,\beta)\,\bar{\kappa}^{(1)}_{T_\beta}(\alpha_1) + \big(\alpha_1 + \omega_2(\alpha_1,\beta)\big)\frac{\beta}{\psi_1(\beta)} - \beta = 0$$

(recalling expression for q_{T_β}). Equivalently,

$$\bar{\kappa}^{(1)}_{T_\beta}(\alpha_1) = \frac{\beta}{r\omega_2(\alpha_1,\beta)} - \left(\frac{\alpha_1}{r\omega_2(\alpha_1,\beta)} + \frac{1}{r}\right)\frac{\beta}{\psi_1(\beta)}.$$

This can now be inserted into $\kappa_{T_\beta}(\boldsymbol{\alpha})$:

$$\kappa_{T_\beta}(\boldsymbol{\alpha}) = \frac{1}{\varphi(\boldsymbol{\alpha}) - \beta}\left(\frac{\beta\alpha_2}{\omega_2(\alpha_1,\beta)} - \left(\frac{\alpha_1\alpha_2}{\omega_2(\alpha_1,\beta)} + \alpha_2\right)\frac{\beta}{\psi_1(\beta)} + \frac{(\alpha_1+\alpha_2)\beta}{\psi_1(\beta)} - \beta\right).$$

Bivariate case: derivation of transform, ctd.

After some rearranging:

Theorem

For any $\boldsymbol{\alpha} \geqslant 0$ *and* $\beta > 0$,

$$\kappa_{T_\beta}(\boldsymbol{\alpha}) = \frac{\alpha_1 - \psi_1(\beta)}{\varphi(\boldsymbol{\alpha}) - \beta} \, \frac{\beta}{\psi_1(\beta)} \, \frac{\omega_2(\alpha_1, \beta) - \alpha_2}{\omega_2(\alpha_1, \beta)}.$$

Check: $\boldsymbol{\alpha} = 0$ yields 1, as desired.

Higher dimensional case

Next goal: recursively solve the case of $d \in \{3, 4, \dots\}$ net cumulative claim processes:

- Claims arrive simultaneously in all d dimensions, according to Poisson process with rate λ.
- Claims $\boldsymbol{B}_1, \boldsymbol{B}_2, \dots$ are d-dimensional, componentwise non-negative i.i.d. random vectors, distributed as generic random vector $\boldsymbol{B}$. Following almost-sure ordering applies:

$$\mathbb{P}\big(B^{(1)} \geqslant B^{(2)} \geqslant \cdots \geqslant B^{(d)}\big) = 1.$$

- Premium rate is, for all net cumulative claim processes, equal to r.

Define $\varphi(\boldsymbol{\alpha})$ as before, with $b(\boldsymbol{\alpha})$ the d-dimensional LST of $\boldsymbol{B}$.

Higher dimensional case

Objective: find transform of random vector $\boldsymbol{Q}(t)$, with $\boldsymbol{Q}(0) = 0$:

$$\kappa_t(\boldsymbol{\alpha}) := \mathbb{E}\, e^{-\boldsymbol{\alpha}^\top \boldsymbol{Q}(t)},$$

evaluated at exponentially distributed time T_β.

Central objects: with $\boldsymbol{x}_{[i]} := (x_1, \ldots, x_i)$ for $i \in \{1, \ldots, d\}$,

$$\begin{aligned}
\bar{\kappa}_t^{(i)}(\boldsymbol{\alpha}_{[i]}) &:= \mathbb{E}\, e^{-\boldsymbol{\alpha}^\top \boldsymbol{Q}(t)} 1\{\boldsymbol{Q}_{[i]}(t) > 0, Q_{i+1}(t) = \ldots = Q_d(t) = 0\} \\
&= \mathbb{E}\, e^{-\boldsymbol{\alpha}_{[i]}^\top \boldsymbol{Q}_{[i]}(t)} 1\{\boldsymbol{Q}_{[i]}(t) > 0, Q_{i+1}(t) = \ldots = Q_d(t) = 0\} \\
&= \mathbb{E}\, e^{-\boldsymbol{\alpha}_{[i]}^\top \boldsymbol{Q}_{[i]}(t)} 1\{\boldsymbol{Q}_{[i]}(t) > 0, Q_{i+1}(t) = 0\},
\end{aligned}$$

where last equality is due to ordering $Q_1(t) \geqslant \ldots \geqslant Q_d(t)$.

Higher dimensional case, ctd.

From bivariate case, we know

$$\bar{\kappa}^{(1)}_{T_\beta}(\boldsymbol{\alpha}_{[1]}) = \mathbb{E}\, e^{-\alpha_{[\mathbf{1}]}^\top Q_{[\mathbf{1}]}(T_\beta)} 1\{Q_{[1]}(T_\beta) > 0, Q_2(T_\beta) = 0\}$$
$$= \mathbb{E}\, e^{-\alpha_{\mathbf{1}} Q_{\mathbf{1}}(T_\beta)} 1\{Q_1(T_\beta) > 0, Q_2(T_\beta) = 0\}.$$

In addition, in Chapter 1 we found

$$\bar{\kappa}^{(0)}_{T_\beta}(\boldsymbol{\alpha}_{[0]}) = q_{T_\beta} := \mathbb{P}(Q_1(T_\beta) = \cdots = Q_d(T_\beta) = 0) = \mathbb{P}(Q_1(T_\beta) = 0).$$

Higher dimensional case, ctd.

Following same procedure as in bivariate case, with each all-ones vector 1 used in the following expression having appropriate dimension,

$$\kappa_{T_\beta}(\boldsymbol{\alpha}) = \frac{r\sum_{i=0}^{d-1}\left(1^\top\boldsymbol{\alpha} - 1^\top\boldsymbol{\alpha}_{[i]}\right)\bar{\kappa}_{T_\beta}^{(i)}(\boldsymbol{\alpha}_{[i]}) - \beta}{\varphi(\boldsymbol{\alpha}) - \beta}.$$

Idea: recursively identify the unknown functions in numerator: supposing that expressions for

$$\bar{\kappa}_{T_\beta}^{(0)}(\boldsymbol{\alpha}_{[0]}), \bar{\kappa}_{T_\beta}^{(1)}(\boldsymbol{\alpha}_{[1]}), \ldots, \bar{\kappa}_{T_\beta}^{(d-2)}(\boldsymbol{\alpha}_{[d-2]})$$

are available, we point out how to determine $\bar{\kappa}_{T_\beta}^{(d-1)}(\boldsymbol{\alpha}_{[d-1]})$.

Higher dimensional case, ctd.

Fixing $\boldsymbol{\alpha}_{[d-1]}$ and β, using same argumentation as before, we can find a unique α_d (in a certain region) such that $\varphi(\boldsymbol{\alpha}) - \beta = 0$; denote this by $\omega_d(\boldsymbol{\alpha}_{[d-1]}, \beta)$.

Any such root of denominator should be root of numerator as well. By some algebra recursive relation

$$\bar{\kappa}^{(d-1)}_{T_\beta}(\boldsymbol{\alpha}_{[d-1]}) = \frac{\beta}{r\omega_d(\boldsymbol{\alpha}_{[d-1]}, \beta)} - \sum_{i=0}^{d-2} \left(\frac{1^\top \boldsymbol{\alpha}_{[d-1]} - 1^\top \boldsymbol{\alpha}_{[i]}}{\omega_d(\boldsymbol{\alpha}_{[d-1]}, \beta)} + 1 \right) \bar{\kappa}^{(i)}_{T_\beta}(\boldsymbol{\alpha}_{[i]})$$

follows.

Higher dimensional case, ctd.

By some additional calculus following result is derived. Here $\omega_j(\boldsymbol{\alpha}_{[j-1]}, \beta)$ is solution for α_j in equation $\varphi(\boldsymbol{\alpha}_{[j]}, 0) - \beta = 0$ (with vector 0 being of dimension $d - j$), for given values of $\boldsymbol{\alpha}_{[j-1]}$ and β.

Theorem

For any $\boldsymbol{\alpha} \geqslant 0$ *and* $\beta > 0$,

$$\kappa_{T_\beta}(\boldsymbol{\alpha}) = \frac{\alpha_1 - \psi_1(\beta)}{\varphi(\boldsymbol{\alpha}) - \beta} \frac{\beta}{\psi_1(\beta)} \prod_{j=2}^{d} \frac{\omega_j(\boldsymbol{\alpha}_{[j-1]}, \beta) - \alpha_j}{\omega_j(\boldsymbol{\alpha}_{[j-1]}, \beta)}.$$

Higher dimensional case, ctd.

There is alternative way to identify transform; see last part of Section 7.3.

Also yields explicit expression for $\bar{\kappa}^{(i)}_{T_\beta}(\boldsymbol{\alpha}_{[i]})$. With

$$\Xi_i(\boldsymbol{\alpha},\beta) := -\frac{\alpha_1-\psi_1(\beta)}{r\omega_{i+1}(\boldsymbol{\alpha}_{[i]},\beta)}\frac{\beta}{\psi_1(\beta)}\prod_{j=2}^{i}\frac{\omega_j(\boldsymbol{\alpha}_{[j-1]},\beta)-\alpha_j}{\omega_j(\boldsymbol{\alpha}_{[j-1]},\beta)},$$

we find:

Corollary

For any $\boldsymbol{\alpha}_{[i]} \geqslant 0$ and $\beta > 0$,

$$\bar{\kappa}^{(i)}_{T_\beta}(\boldsymbol{\alpha}_{[i]}) = \Xi_i(\boldsymbol{\alpha},\beta) - \Xi_{i-1}(\boldsymbol{\alpha},\beta).$$

Tandem system

Model: tandem queueing network fed by a compound Poisson process $Y(t)$ with arrival rate $\lambda > 0$. The i.i.d. service requirements $B_1, B_2, \ldots$ are distributed as rv B with LST $b(\alpha)$.
Consider d queues in series, with (constant) service rates $c_1, \ldots, c_d$ that are non-increasing (i.e., $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_d$).
Output of i-th queue is continuously fed into $(i+1)$-st queue, for $i = 1, \ldots, d-1$; no external input arrives.

Framework is seemingly different from the one discussed earlier, but joint workload distribution immediately follows from earlier results.

Tandem system, ctd.

Idea: above tandem network fits into our setup, as follows.
$Q_i(t)$: workload in the i-th queue, with $i = 1, \ldots, d$, at time $t \geqslant 0$; assume system starts empty at time 0. Recall that workload in first queue obeys

$$Q_1(t) = (Y(t) - c_1 t) - \inf_{s\in[0,t]} (Y(s) - c_1 s).$$

Now consider $Q_1(t) + Q_2(t)$, which is only affected by service rate c_2 (not by c_1). This means that

$$Q_1(t) + Q_2(t) = (Y(t) - c_2 t) - \inf_{s\in[0,t]} (Y(s) - c_2 s).$$

Extending this argument, we obtain for any $i = 1, \ldots, d$,

$$Q^{(i)}(t) := \sum_{j=1}^{i} Q_j(t) = (Y(t) - c_i t) - \inf_{s\in[0,t]} (Y(s) - c_i s).$$

Tandem system, ctd.

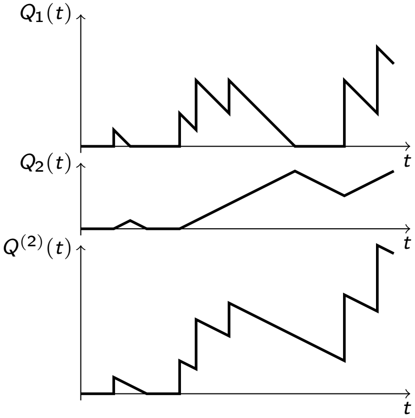

Figure: Tandem queueing processes $Q_1(t)$ and $Q_2(t)$, and sum $Q^{(2)}(t)$. $Q_1(t)$ is M/G/1 queue with drain rate c_1 and $Q^{(2)}(t)$ is M/G/1 queue with drain rate c_2. While not empty, $Q_2(t)$ increases at rate $c_1 - c_2$ and decreases at rate c_2.

Tandem system, ctd.

Observe $Q^{(i)}(t)/c_i$ can be seen as workload in queue fed by compound Poisson process with arrival rate λ and i.i.d. service requirements distributed as B/c_i, emptied at a unit rate. But because

$$\mathbb{P}\big(B/c_d \geqslant B/c_{d-1} \geqslant \cdots \geqslant B/c_1\big) = 1,$$

we can apply earlier results to describe joint distribution of these d workloads (with indices $1, \ldots, d$ being swapped), and hence also of original d workloads $Q_1(t), \ldots, Q_d(t)$. See Theorem 7.3.

Gerber-Shiu metrics

So far: focus on joint ruin probability. Now: joint distribution of ruin times of both insurance firms, together with corresponding undershoots and overshoots, i.e., so-called (multivariate) *Gerber-Shiu metrics.*

Here we do bivariate case, but can be extended to higher dimensions.

Abbreviate $\boldsymbol{u} = (u_1, u_2)^\top \in [0, \infty)^2$, $\boldsymbol{Y}(t) = (Y_1(t), Y_2(t))^\top$, $\boldsymbol{\tau}(\boldsymbol{u}) = (\tau_1(u_1), \tau_2(u_2))^\top$, and

$$\boldsymbol{Y}(\boldsymbol{\tau}(\boldsymbol{u})-) := \begin{pmatrix} Y_1(\tau_1(u_1)-) \\ Y_2(\tau_2(u_2)-) \end{pmatrix}, \quad \boldsymbol{Y}(\boldsymbol{\tau}(\boldsymbol{u})) := \begin{pmatrix} Y_1(\tau_1(u_1)) \\ Y_2(\tau_2(u_2)) \end{pmatrix};$$

here $\tau_i(u_i)$ is ruin time corresponding to net cumulative claim process $Y_i(t)$, i.e., smallest $t \geqslant 0$ such that $Y_i(t) \geqslant u_i$.

Gerber-Shiu metrics, ctd.

Object of study, for $\boldsymbol{u} \geqslant 0$ and $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \geqslant 0$, $\boldsymbol{\gamma}_3 \leqslant 0$,

$$\begin{aligned} p(\boldsymbol{u}) &\equiv p(\boldsymbol{u}, \beta, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3) \\ &:= \mathbb{E}\Big(e^{-\boldsymbol{\gamma}_1^\top \boldsymbol{\tau}(\boldsymbol{u}) - \boldsymbol{\gamma}_2^\top (\boldsymbol{u} - \boldsymbol{Y}(\boldsymbol{\tau}(\boldsymbol{u})-)) - \boldsymbol{\gamma}_3^\top (\boldsymbol{u} - \boldsymbol{Y}(\boldsymbol{\tau}(\boldsymbol{u})))} 1\{\boldsymbol{\tau}(\boldsymbol{u}) \leqslant T_\beta \, \mathbf{1}\}\Big), \end{aligned}$$

where $\boldsymbol{\gamma}_i = (\gamma_{i1}, \gamma_{i2})^\top$ for $i = 1, 2, 3$.

We analyze $p(\boldsymbol{u})$ through (nine-fold) transform, where $\boldsymbol{\alpha} \geqslant 0$ and $\beta > 0$,

$$\pi(\boldsymbol{\alpha}) \equiv \pi(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3) := \int_0^\infty \int_0^\infty e^{-\boldsymbol{\alpha}^\top \boldsymbol{u}} \, p(\boldsymbol{u}, \beta, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3) \, du_1 \, du_2.$$

Define univariate counterparts of $p(\boldsymbol{u})$:

$$\begin{aligned} p_i(u) &\equiv p(u, \beta, \gamma_{1i}, \gamma_{2i}, \gamma_{3i}) \\ &:= \mathbb{E}\Big(e^{-\gamma_{\mathbf{1}i}\tau_i(u) - \gamma_{\mathbf{2}i}(u - Y_i(\tau_i(u)-)) - \gamma_{\mathbf{3}i}(u - Y_i(\tau_i(u)))} 1\{\tau_i(u) \leqslant T_\beta\}\Big). \end{aligned}$$

Gerber-Shiu metrics, ctd.

Equation our analysis is based on, as $\Delta t \downarrow 0$, cf. Exercise 1.2:

$$\begin{aligned} p(\boldsymbol{u}) \; &= e^{-\boldsymbol{\gamma_1}^\top 1\,\Delta t}\Big(\lambda\,\Delta t \int_{v_1=0}^{u_1}\int_{v_2=0}^{u_2} p(\boldsymbol{u}-\boldsymbol{v})\,\mathbb{P}(\boldsymbol{B}\in d\boldsymbol{v}) + \\ &\qquad \lambda\,\Delta t \int_{v_1=0}^{u_1}\int_{v_2=u_2}^{\infty} p_1(u_1-v_1)\,e^{-\gamma_{22}u_2-\gamma_{32}(u_2-v_2)}\,\mathbb{P}(\boldsymbol{B}\in d\boldsymbol{v}) + \\ &\qquad \lambda\,\Delta t \int_{v_1=u_1}^{\infty}\int_{v_2=0}^{u_2} p_2(u_2-v_2)\,e^{-\gamma_{21}u_1-\gamma_{31}(u_1-v_1)}\,\mathbb{P}(\boldsymbol{B}\in d\boldsymbol{v}) + \\ &\qquad \lambda\,\Delta t \int_{v_1=u_1}^{\infty}\int_{v_2=u_2}^{\infty} e^{-\boldsymbol{\gamma_2}^\top\boldsymbol{u}-\boldsymbol{\gamma_3}^\top(\boldsymbol{u}-\boldsymbol{v})}\,\mathbb{P}(\boldsymbol{B}\in d\boldsymbol{v}) + \\ &\qquad \big(1-(\lambda+\beta)\Delta t\big)\,p(\boldsymbol{u}+r\,1\Delta t)\Big) + o(\Delta t). \end{aligned}$$

Gerber-Shiu metrics, ctd.

Standard procedure: subtract $p(\boldsymbol{u} + r\,1\Delta t)$ from both sides, divide by Δt, and let $\Delta t \downarrow 0$:

$$-r\Big(\frac{\partial}{\partial u_1}p(\boldsymbol{u}) + \frac{\partial}{\partial u_2}p(\boldsymbol{u})\Big) = \lambda \int_0^{u_1}\int_0^{u_2} p(\boldsymbol{u}-\boldsymbol{v})\,\mathbb{P}(\boldsymbol{B}\in d\boldsymbol{v}) +$$

$$\lambda \int_0^{u_1}\int_{u_2}^{\infty} p_1(u_1 - v_1)\,e^{-\gamma_{22}u_2-\gamma_{32}(u_2-v_2)}\,\mathbb{P}(\boldsymbol{B}\in d\boldsymbol{v}) +$$

$$\lambda \int_{u_1}^{\infty}\int_0^{u_2} p_2(u_2 - v_2)\,e^{-\gamma_{21}u_1-\gamma_{31}(u_1-v_1)}\,\mathbb{P}(\boldsymbol{B}\in d\boldsymbol{v}) +$$

$$\lambda \int_{u_1}^{\infty}\int_{u_2}^{\infty} e^{-\boldsymbol{\gamma}_2^\top\boldsymbol{u}-\boldsymbol{\gamma}_3^\top(\boldsymbol{u}-\boldsymbol{v})}\,\mathbb{P}(\boldsymbol{B}\in d\boldsymbol{v}) - \big(1^\top\boldsymbol{\gamma}_1 + \lambda + \beta\big)\,p(u).$$

Gerber-Shiu metrics, ctd.

Compute transform with respect to $\boldsymbol{u}$: multiply full equation by $e^{-\boldsymbol{\alpha}^\top \boldsymbol{u}}$ and integrate over non-negative u_1 and u_2. RHS becomes:

$$\left(\lambda b(\boldsymbol{\alpha}) - 1^\top \boldsymbol{\gamma}_1 - \lambda - \beta\right) \pi(\boldsymbol{\alpha}) + \lambda \zeta(\boldsymbol{\alpha}),$$

where

$$\zeta(\boldsymbol{\alpha}) := \ \pi_1(\alpha_1)\, \frac{b(\alpha_1, -\gamma_{32}) - b(\alpha_1, \alpha_2 + \gamma_{22})}{\alpha_2 + \gamma_{22} + \gamma_{32}} +$$
$$\pi_2(\alpha_2)\, \frac{b(-\gamma_{31}, \alpha_2) - b(\alpha_1 + \gamma_{21}, \alpha_2)}{\alpha_1 + \gamma_{21} + \gamma_{31}} +$$
$$\frac{b(-\gamma_{31}, -\gamma_{32}) - b(-\gamma_{31}, \alpha_2 + \gamma_{22}) - b(\alpha_1 + \gamma_{21}, -\gamma_{32}) + b(\alpha_1 + \gamma_{21}, \alpha_2 + \gamma_{22})}{(\alpha_1 + \gamma_{21} + \gamma_{31})(\alpha_2 + \gamma_{22} + \gamma_{32})}.$$

LHS becomes:

$$-r\, 1^\top \boldsymbol{\alpha}\, \pi(\boldsymbol{\alpha}) + r\pi_1^\circ(\alpha_2) + r\pi_2^\circ(\alpha_1),$$

where

$$\pi_1^\circ(\alpha) := \int_0^\infty p(0, u)\, e^{-\alpha u}\, du, \quad \pi_2^\circ(\alpha) := \int_0^\infty p(u, 0)\, e^{-\alpha u}\, du.$$

Gerber-Shiu metrics, ctd.

Proposition

For any $\boldsymbol{\alpha} \geqslant 0$, $\beta > 0$, $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \geqslant 0$, $\boldsymbol{\gamma}_3 \leqslant 0$,

$$\pi(\boldsymbol{\alpha}) = \frac{r(\pi_1^\circ(\alpha_2) + \pi_2^\circ(\alpha_1)) - \lambda\zeta(\boldsymbol{\alpha})}{\varphi(\boldsymbol{\alpha}) - 1^\top\boldsymbol{\gamma}_1 - \beta}.$$

Left: identification of functions $\pi_i^\circ(\alpha)$. Key idea: ordering $Y_1(t) \geqslant Y_2(t)$ can be used to evaluate $\pi_1^\circ(\alpha)$, where crucial role is played by $\tau_1(0) \leqslant \tau_2(u)$ for all $u \geqslant 0$. Then, by Lemma:

$$\pi_2^\circ(\alpha) = -\pi_1^\circ\big(\omega_2(\alpha, 1^\top\boldsymbol{\gamma}_1 + \beta)\big) + \frac{\lambda}{r}\,\zeta\big(\alpha, \omega_2(\alpha, 1^\top\boldsymbol{\gamma}_1 + \beta)\big).$$

Gerber-Shiu metrics, ctd.

Define

$$\boldsymbol{W}(u) := (Y_2(\tau_1(u)), B_2^\circ(u))^\top;$$

$B_2^\circ(u)$ is claim size in $Y_2(t)$ at ruin time $\tau_1(u)$ corresponding to $Y_1(t)$.

We need, with $\mathbb{I}(u, d\boldsymbol{w}) := 1\{\tau_1(u) \leqslant T_\beta, \boldsymbol{W}(u) \in d\boldsymbol{w}\}$,

$$\bar{p}_1(u, d\boldsymbol{w}) := \mathbb{E}\Big(e^{-1^\top \boldsymbol{\gamma_1}\tau_1(u) - \gamma_{21}(u - Y_1(\tau_1(u)-)) - \gamma_{31}(u - Y_1(\tau_1(u)))}\,\mathbb{I}(u, d\boldsymbol{w})\Big).$$

Key identity (use that $\tau_1(0) \leqslant \tau_2(u)$ for all $u \geqslant 0$!):

$$p(0,u) = \int_{w_1=u}^{\infty}\int_{w_2=0}^{\infty} \bar{p}_1(0, d\boldsymbol{w})\, e^{-\gamma_{22}(u - w_1 + w_2) - \gamma_{32}(u - w_1)} + \int_{w_1=-\infty}^{u}\int_{w_2=0}^{\infty} \bar{p}_1(0, d\boldsymbol{w})\, p_2(u - w_1).$$

First scenario: $Y_2(t)$ first exceeds u at $\tau_1(0)$ (i.e., $\tau_1(0) = \tau_2(u)$).
Second scenario: u is not yet exceeded by $Y_2(t)$ at time $\tau_1(0)$ (i.e., $\tau_1(0) < \tau_2(u)$). See book for further explanation.

Gerber-Shiu metrics, ctd.

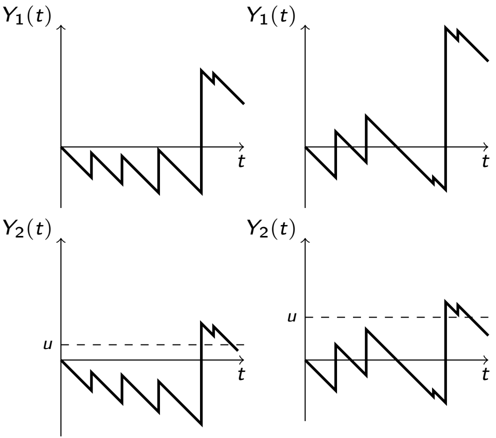

Figure: Processes $Y_1(t)$ and $Y_2(t)$ such that $Y_1(t) \geqslant Y_2(t)$ for all $t \geqslant 0$. Left panels: scenario of $(Y_1(t), Y_2(t))$ in which $\tau_1(0) = \tau_2(u)$. Right panels: scenario of $(Y_1(t), Y_2(t))$ in which $\tau_1(0) < \tau_2(u)$.

Gerber-Shiu metrics, ctd.

Define, for $\boldsymbol{\delta} \in \mathbb{R}^2$,

$$\xi(\boldsymbol{\delta}) := \mathbb{E}\big(e^{-\mathbf{1}^\top \boldsymbol{\gamma}_{\mathbf{1}} \tau_{\mathbf{1}}(0) + \gamma_{\mathbf{21}} Y_{\mathbf{1}}(\tau_{\mathbf{1}}(0)-) + \gamma_{\mathbf{31}} Y_{\mathbf{1}}(\tau(0)) - \delta_{\mathbf{1}} Y_{\mathbf{2}}(\tau_{\mathbf{1}}(0)) - \delta_{\mathbf{2}} B^{\circ}_{\mathbf{2}}(0)} 1\{\tau_1(0) \leqslant T_\beta\}$$

Then,

$$\begin{aligned} \pi_1^{\circ}(\alpha) = & \int_0^\infty e^{-\alpha u} \int_{w_{\mathbf{1}}=u}^\infty \int_{w_{\mathbf{2}}=0}^\infty \bar{p}_1(0, d\boldsymbol{w})\, e^{-\gamma_{\mathbf{22}}(u - w_{\mathbf{1}} + w_{\mathbf{2}}) - \gamma_{\mathbf{32}}(u - w_{\mathbf{1}})}\, du \, + \\ & \int_0^\infty e^{-\alpha u} \int_{w_{\mathbf{1}}=-\infty}^u \int_{w_{\mathbf{2}}=0}^\infty \bar{p}_1(0, d\boldsymbol{w})\, p_2(u - w_1)\, du \\ = & \frac{\xi(-\gamma_{22} - \gamma_{32}, \gamma_{22}) - \xi(\alpha, \gamma_{22})}{\alpha + \gamma_{22} + \gamma_{32}} + \xi(\alpha, 0)\, \pi_2(\alpha); \end{aligned}$$

second equality follows by swapping the order of the integrals and standard calculus.

Gerber-Shiu metrics, ctd.

To evaluate this expression, we study

$$\check{p}_1(u) \equiv \check{p}_1(u, \boldsymbol{\delta})$$
$$:= \mathbb{E}\Big(e^{-\mathbf{1}^\top \boldsymbol{\gamma}_{\mathbf{1}} \tau_{\mathbf{1}}(u) - \gamma_{\mathbf{21}}(u - Y_{\mathbf{1}}(\tau_{\mathbf{1}}(u)-)) - \gamma_{\mathbf{31}}(u - Y_{\mathbf{1}}(\tau(u))) - \boldsymbol{\delta}^\top \boldsymbol{W}(u)} 1\{\tau_1(u) \leqslant T_\beta\}\Big).$$

Due to $\xi(\boldsymbol{\delta}) = \check{p}_1(0, \boldsymbol{\delta})$, if we have access to $\check{p}_1(0, \boldsymbol{\delta})$, then by inserting specific values for δ_1 and δ_2, we can compute all terms.

Gerber-Shiu metrics, ctd.

As in Exercise 1.2, determine transform of $\check{p}_1(u)$. First,

$$\check{p}_1(u) = e^{-\mathbf{1}^\top \boldsymbol{\gamma}_1 \, \Delta t + r\delta_1 \, \Delta t}\Big(\lambda \, \Delta t \int_{v_1=0}^{u} \int_{v_2=0}^{\infty} \mathbb{P}(\boldsymbol{B} \in d\boldsymbol{v}) \, \check{p}_1(u - v_1) \, e^{-\delta_1 v_2} + \lambda \, \Delta t \int_{v_1=u}^{\infty} \int_{v_2=0}^{\infty} \mathbb{P}(\boldsymbol{B} \in d\boldsymbol{v}) \, e^{-\gamma_{21} u} \, e^{-\gamma_{31}(u-v_1)} \, e^{-\mathbf{1}^\top \boldsymbol{\delta} \, v_2} + (1 - \lambda \, \Delta t - \beta \, \Delta t) \, \check{p}_1(u + r \, \Delta t) \Big).$$

Subtract $\check{p}_1(u + r\,\Delta t)$ from both sides, divide by Δt, and let $\Delta t \downarrow 0$, so as to obtain integro-differential equation. Taking transforms,

$$\check{\pi}_1(\alpha) = \frac{1}{\varphi(\alpha, \delta_1) - \mathbf{1}^\top \boldsymbol{\gamma}_1 - \beta} \left(r \, \check{p}_1(0) - \lambda \frac{b(-\gamma_{31}, \mathbf{1}^\top \boldsymbol{\delta}) - b(\alpha + \gamma_{21}, \mathbf{1}^\top \boldsymbol{\delta})}{\alpha + \gamma_{21} + \gamma_{31}} \right),$$

with

$$\check{\pi}_1(\alpha) := \int_0^\infty e^{-\alpha u} \, \check{p}_1(u) \, du.$$

Gerber-Shiu metrics, ctd.

Next step: determine $\check{p}_1(0)$. Any value of α (with non-negative real part, that is) for which $\varphi(\alpha, \delta_1) - 1^\top \boldsymbol{\gamma}_1 - \beta$ equals zero, term between brackets in $\check{\pi}_1(\alpha)$ should equal zero as well.

Using compact notation $\alpha^\circ \equiv \alpha^\circ(\beta, \boldsymbol{\gamma}_1, \delta_1) := \psi_1(1^\top \boldsymbol{\gamma}_1 + \beta)$, with $\beta \mapsto \psi_1(\beta)$ denoting the right-inverse of $\alpha \mapsto \varphi(\alpha, \delta_1)$,

$$\check{p}_1(0, \boldsymbol{\delta}) = \xi(\boldsymbol{\delta}) = \frac{\lambda}{r} \frac{b(-\gamma_{31}, 1^\top \boldsymbol{\delta}) - b(\alpha^\circ + \gamma_{21}, 1^\top \boldsymbol{\delta})}{\alpha^\circ + \gamma_{21} + \gamma_{31}}.$$

We found all ingredients that allow evaluation of $\pi_1^\circ(\alpha)$.

Gerber-Shiu metrics, ctd.

Theorem

For any $\boldsymbol{\alpha} \geqslant 0$*,* $\beta > 0$*,* $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \geqslant 0$*,* $\boldsymbol{\gamma}_3 \leqslant 0$*,*

$$\pi(\boldsymbol{\alpha}) = \frac{r(\pi_1^\circ(\alpha_2) + \pi_2^\circ(\alpha_1)) - \lambda\zeta(\boldsymbol{\alpha})}{\varphi(\boldsymbol{\alpha}) - 1^\top\boldsymbol{\gamma}_1 - \beta},$$

where

$$\pi_2^\circ(\alpha) = -\pi_1^\circ\big(\omega_2(\alpha, 1^\top\boldsymbol{\gamma}_1 + \beta)\big) + \frac{\lambda}{r}\,\zeta\big(\alpha, \omega_2(\alpha, 1^\top\boldsymbol{\gamma}_1 + \beta)\big)$$

and $\pi_1^\circ(\alpha)$ *as determined above.*

CHAPTER VIII: ARRIVAL PROCESSES WITH CLUSTERING

Arrival processes with clustering: main ideas

This chapter: CL model driven by claim arrival process with randomly fluctuating rate.

Arrival rate is *stochastic process*, evolving as

- $M/G/\infty$ queue (to do justice to fluctuating number of clients);
- shot-noise process (to model impact of catastrophic events);
- Hawkes process (to model effect of claims triggering additional claims).

Objective: determine, in light-tailed context, decay rate of ruin probability.

The proofs rely either on change-of-measure, or on large deviations argumentation.

Arrival processes with clustering: main ideas, ctd.

Exact analysis of $p(u)$ or $p(u, T_\beta)$ is prohibitively difficult. Therefore: asymptotics of $p(u)$.

Relevant in analysis: *Limiting Laplace exponent* $\Phi(\alpha)$. With $Y(t)$ net cumulative claim process,

$$\Phi(\alpha) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\, e^{-\alpha Y(t)}.$$

Assume net-profit condition holds:

$$\lim_{t\to\infty} \frac{\mathbb{E}\, Y(t)}{t} = -\Phi'(0) < 0.$$

Arrival processes with clustering: main ideas, ctd.

$\Phi(-\theta)$: limiting moment generating function.

Other relevant function: *Legendre transform* $I(a)$. For $a \in \mathbb{R}$,

$$I(a) := \sup_{\theta>0} \big(\theta a - \Phi(-\theta)\big),$$

which is non-negative and convex, and attains its minimal value 0 at $a = -\Phi'(0)$; see Exercise 8.1.

Arrival processes with clustering: main ideas, ctd.

For all three arrival processes, we prove

$$\lim_{u\to\infty} \frac{1}{u} \log p(u) = -\theta^\star,$$

where $\theta^\star > 0$ is such that $\Phi(-\theta^\star) = 0$.

Strategy: prove that $-\theta^\star$ is lower bound (follows easily), and prove that $-\theta^\star$ is upper bound (way harder).

Arrival processes with clustering: main ideas, ctd.

Lower bound: as in large deviations based approach (Section 2.2):

- Observe: for any $T > 0$, $p(u) = \mathbb{P}(\bar{Y}(\infty) \geqslant u) \geqslant \mathbb{P}(Y(Tu) \geqslant u)$.
- Hence, for all $T, u > 0$,

$$\frac{1}{u}\log p(u) \geqslant \frac{T}{Tu}\log \mathbb{P}\left(\frac{Y(Tu)}{Tu} \geqslant \frac{1}{T}\right).$$

- Consequently, for all $T > 0$,

$$\liminf_{u\to\infty} \frac{1}{u}\log p(u) \geqslant -T\, I(1/T).$$

  (as the increments are now not i.i.d., instead of *Cramér's theorem*, the *Gärtner-Ellis* theorem needs to be used).
- Lower bound applies to any $T > 0$. Hence,

$$\liminf_{u\to\infty} \frac{1}{u}\log p(u) \geqslant -I^\star := -\inf_{T>0} TI(1/T).$$

- Then, as in Section 2.2, $I^\star = \theta^\star$.

Arrival processes with clustering: main ideas, ctd.

Large deviations results allow for appealing interpretation.
Denote $T^\star := \arg\inf_{T>0} TI(1/T)$.

Then $\Delta^\star := 1/T^\star$ can be interpreted as 'cheapest' slope to reach high level. Given high level u is exceeded (rare event!), the most likely way is 'roughly linear' with slope $\Delta^\star$.

Likewise, $T^\star u$ is proxy for typical time it takes to exceed level u.

Arrival processes with clustering: main ideas, ctd.

Upper bound: Considerably harder!

We consider three arrival processes; proofs rely on techniques developed in Section 2.2:

- ◦ for model with M/G/∞ driven arrivals we use proof based on a change-of-measure,
- ◦ whereas for shot-noise and Hawkes driven arrivals we rely on large-deviations based argumentation.

M/G/∞ driven arrivals

Model:

- New clients arrive according to a Poisson process with rate $\nu > 0$. They stay i.i.d. times in system, with $d(\alpha)$ LST of generic sojourn time D.
  Number of clients simultaneously present: M/G/∞ system.
  Stationary distribution is Poisson with parameter $\nu \, \mathbb{E}D$.
- While in system each client generates i.i.d. claims with LST $b(\alpha)$ according to Poisson process with rate λ.
- Premiums are generated at constant rate r (by full population, being of fluctuating size, that is).

The claim arrival rate is thus following stochastic process $\Lambda(t)$ that is proportional to the number of clients in M/G/∞ queue.

M/G/∞ driven arrivals, ctd.

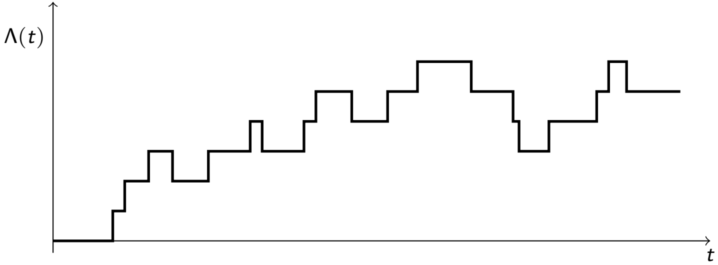

Figure: Arrival rate process $\Lambda(t)$ in M/G/∞ case.

M/G/∞ driven arrivals, ctd.

Net profit condition:

$$\lambda\,(\nu\mathbb{E}D)\cdot\mathbb{E}B < r,$$

(Interpretation?).

Proposition

As $t\to\infty$,

$$\frac{1}{t}\log\mathbb{E}\,e^{-\alpha Y(t)}\to\Phi(\alpha)=r\alpha-\nu+\nu\,d\Big(\,\lambda(1-b(\alpha))\,\Big).$$

M/G/∞ driven arrivals, ctd.

Proof. Number of client arrivals in $[0,t)$ is Poisson with mean νt. Well-known: given the number of arrivals, each of them enters at a position that is uniformly distributed on $(0,t)$.

Hence,

$$\frac{1}{t}\log \mathbb{E}\, e^{-\alpha Y(t)} = r\alpha + \frac{1}{t}\log\sum_{i=0}^{\infty} e^{-\nu t}\frac{(\nu t)^i}{i!}\big(Z_t(\alpha)\big)^i = r\alpha - \nu + \nu Z_t(\alpha),$$

where

$$Z_t(\alpha) := \frac{1}{t}\Bigg(\int_0^t\int_0^u \mathbb{P}(D\in ds)\sum_{j=0}^{\infty} e^{-\lambda s}\frac{(\lambda s)^j}{j!}\big(b(\alpha)\big)^j\,du + \int_0^t \mathbb{P}(D\geqslant u)\sum_{j=0}^{\infty} e^{-\lambda u}\frac{(\lambda u)^j}{j!}\big(b(\alpha)\big)^j\,du\Bigg).$$

(Check!).

M/G/∞ driven arrivals, ctd.

Simplifies to

$$\frac{1}{t}\left(\int_0^t\int_0^u \mathbb{P}(D\in ds)e^{-\lambda s(1-b(\alpha))}\,du+\int_0^t \mathbb{P}(D\geqslant u)e^{-\lambda u(1-b(\alpha))}\,du\right).$$

First term: clients who have left by time t. Second term: clients who are still present at time t.

Left: computation of the limit of $Z_t(\alpha)$ as $t\to\infty$. First term: interchanging integrals gives

$$\begin{aligned}\frac{1}{t}\int_0^t\int_0^u \mathbb{P}(D\in ds)e^{-\lambda s(1-b(\alpha))}\,du &= \int_0^t \frac{t-s}{t}\mathbb{P}(D\in ds)e^{-\lambda s(1-b(\alpha))}\\ &\to d\big(\,\lambda(1-b(\alpha))\,\big).\end{aligned}$$

Second term vanishes.

M/G/∞ driven arrivals, ctd.

Let $\theta^\star > 0$ solve $\Phi(-\theta^\star) = 0$ (implicitly requires both the clients' sojourn times and claim sizes to have light-tailed distributions).

Change of measure: $\Phi_{\mathbb{Q}}(\alpha) = \Phi(\alpha - \theta^\star)$.

We can rewrite, with $\lambda_{\mathbb{Q}} := \lambda b(-\theta^\star)$ and $d_{\mathbb{Q}} := d(\lambda - \lambda_{\mathbb{Q}})$,

$$\begin{aligned}\Phi(\alpha - \theta^\star) &= r(\alpha - \theta^\star) - \nu + \nu\, d\big(\lambda(1 - b(\alpha - \theta^\star))\big)\\ &= r\alpha - \nu\, d_{\mathbb{Q}} + \nu\, d_{\mathbb{Q}} \frac{d\left(\lambda_{\mathbb{Q}}\left(1 - \dfrac{b(\alpha - \theta^\star)}{b(-\theta^\star)}\right) + \lambda - \lambda_{\mathbb{Q}}\right)}{d_{\mathbb{Q}}}.\end{aligned}$$

M/G/∞ driven arrivals, ctd.

Conclude: under new measure $\mathbb{Q}$ process $Y(t)$ is still M/G/∞ driven net cumulative claim process, but now with

- client arrival rate $\nu_{\mathbb{Q}} := \nu\, d_{\mathbb{Q}}$,
- client sojourn times with LST

$$\mathbb{E}_{\mathbb{Q}} e^{-\alpha D} = \frac{d(\alpha + \lambda - \lambda_{\mathbb{Q}})}{d(\lambda - \lambda_{\mathbb{Q}})},$$

- claim arrival rate $\lambda_{\mathbb{Q}}$,
- and claim sizes with LST

$$\mathbb{E}_{\mathbb{Q}} e^{-\alpha B} = \frac{b(\alpha - \theta^\star)}{b(-\theta^\star)}.$$

M/G/∞ driven arrivals, ctd.

Informally, process $Y(t)$ reaching high level u is combined effect of:
(i) higher client arrival rate, (ii) longer client sojourn times, (iii) higher claim arrival rate, and (iv) larger claims.

M/G/∞ driven arrivals, ctd.

Objective: derive upper bound $p(u) \leqslant e^{-\theta^\star u}$. Mimic change-of-measure based approach of Section 2.2.

At moment $\tau(u)$ that $[u, \infty)$ has been reached, we have sampled client interarrival times $\boldsymbol{F} \equiv (F_1, \ldots, F_N)$ and their sojourn times $\boldsymbol{D} \equiv (D_1, \ldots, D_N)$.

For each of the clients, we sample number of claims during their sojourn time, i.e., $\boldsymbol{M} \equiv (M_1, \ldots, M_N)$, where the corresponding arrival epochs are uniformly distributed over their sojourn times.
Claim sizes are

$$\boldsymbol{B} \equiv \left(B_{11}, \ldots, B_{1M_1}, B_{21}, \ldots, B_{2M_2}, \ldots, B_{N1,\ldots,NM_N} \right).$$

Precise sampling procedure: see book.

M/G/∞ driven arrivals, ctd.

Each time random object is sampled: update the likelihood ratio.

Let $\tau(u)$ be stopping time. Is (under $\mathbb{Q}$) finite almost surely. (Why?) As before: N the number of clients that have arrived by time $\tau(u)$.

Hence $p(u)$ equals likelihood ratio

$$\mathbb{E}_{\mathbb{Q}} L(\boldsymbol{F}, \boldsymbol{D}, \boldsymbol{M}, \boldsymbol{B}).$$

Let $f_{\mathbb{P}}(\cdot)$ and $f_{\mathbb{Q}}(\cdot)$ be densities of B under $\mathbb{P}$ and $\mathbb{Q}$, respectively. Likewise, $g_{\mathbb{P}}(\cdot)$ and $g_{\mathbb{Q}}(\cdot)$ are densities of D under $\mathbb{P}$ and $\mathbb{Q}$, respectively.

The likelihood ratio can be decomposed into four factors.

M/G/∞ driven arrivals, ctd.

1. First (L_F) corresponds to client arrivals. Suppose first arrival is at time s, we obtain evident contribution

$$\frac{\nu}{\nu_{\mathbb{Q}}}\frac{e^{-\nu s}}{e^{-\nu_{\mathbb{Q}} s}} = \frac{1}{d_{\mathbb{Q}}}\frac{e^{-\nu s}}{e^{-\nu_{\mathbb{Q}} s}}.$$

For this client we sample its specifics (sojourn time, claim arrival times, claim sizes).
Next client arrival: suppose it is scheduled at (say) t time units from current time, if this leads to a client arrival at $s \in (0, t]$ time units from current time, then we get contribution

$$\frac{1-e^{-\nu t}}{1-e^{-\nu_{\mathbb{Q}} t}}\frac{\nu\, e^{-\nu s}/(1-e^{-\nu t})}{\nu_{\mathbb{Q}}\, e^{-\nu_{\mathbb{Q}} s}/(1-e^{-\nu_{\mathbb{Q}} t})} = \frac{\nu}{\nu_{\mathbb{Q}}}\frac{e^{-\nu s}}{e^{-\nu_{\mathbb{Q}} s}} = \frac{1}{d_{\mathbb{Q}}}\frac{e^{-\nu s}}{e^{-\nu_{\mathbb{Q}} s}};$$

if it does not lead to client arrival before next scheduled event, then contribution is

$$\frac{e^{-\nu t}}{e^{-\nu_{\mathbb{Q}} t}}.$$

M/G/∞ driven arrivals, ctd.

1. Combining the above,

$$L_{\boldsymbol{F}} = e^{(\nu_{\mathbb{Q}} - \nu)\,\tau(u)} \left(\frac{\nu}{\nu_{\mathbb{Q}}}\right)^{N} = e^{(\nu_{\mathbb{Q}} - \nu)\,\tau(u)} \big(\, d_{\mathbb{Q}} \,\big)^{-N}.$$

2. Second contribution, $L_{\boldsymbol{D}}$, corresponds to the sojourn time durations. Check that

$$L_{\boldsymbol{D}} = \prod_{i=1}^{N} \frac{g_{\mathbb{P}}(D_i)}{g_{\mathbb{Q}}(D_i)} = e^{(\lambda - \lambda_{\mathbb{Q}})\sum_{i=1}^{N} D_i} \big(\, d_{\mathbb{Q}} \,\big)^{N}.$$

M/G/∞ driven arrivals, ctd.

3. Third contribution concerns claim arrival times. Under both $\mathbb{P}$ and $\mathbb{Q}$, conditional on number of arrivals, arrival epochs are uniformly distributed, independently of each other (thus not contributing to likelihood ratio). Therefore,

$$L_{\boldsymbol{M}} = \prod_{i=1}^{N} \frac{e^{-\lambda D_i}(\lambda D_i)^{M_i}/M_i!}{e^{-\lambda_{\mathbb{Q}} D_i}(\lambda_{\mathbb{Q}} D_i)^{M_i}/M_i!} = e^{-(\lambda-\lambda_{\mathbb{Q}})\sum_{i=\mathbf{1}}^{N} D_i}\left(\frac{\lambda}{\lambda_{\mathbb{Q}}}\right)^{\sum_{i=\mathbf{1}}^{N} M_i}$$
$$= e^{-(\lambda-\lambda_{\mathbb{Q}})\sum_{i=\mathbf{1}}^{N} D_i}(b(-\theta^\star))^{-M^+},$$

   where $M^+ := \sum_{i=1}^{N} M_i$.

4. Last contribution concerns claim sizes:

$$L_{\boldsymbol{B}} = \prod_{i=1}^{N}\prod_{j=1}^{M_i} \frac{f_{\mathbb{P}}(B_{ij})}{f_{\mathbb{Q}}(B_{ij})} = e^{-\theta^\star B^+}(b(-\theta^\star))^{M^+}, \ \textit{with} \quad B^+ := \sum_{i=1}^{N}\sum_{j=1}^{M_i} B_{ij}.$$

M/G/∞ driven arrivals, ctd.

Since B^+ is sum of the claims issued by time $\tau(u)$,

$$B^+ - r\tau(u) \geqslant Y(\tau(u)) \geqslant u.$$

(Why?) Recalling that $r\theta^\star = \nu_{\mathbb{Q}} - \nu$, we find an upper bound for $p(u)$:

$$p(u) = \mathbb{E}_{\mathbb{Q}} L(\boldsymbol{F}, \boldsymbol{D}, \boldsymbol{M}, \boldsymbol{B}) = \mathbb{E}_{\mathbb{Q}}\big[L_{\boldsymbol{F}} L_{\boldsymbol{D}} L_{\boldsymbol{M}} L_{\boldsymbol{B}}\big]$$
$$= e^{(\nu_{\mathbb{Q}} - \nu)\,\tau(u)} e^{-\theta^\star B^+} \leqslant e^{-\theta^\star u}.$$

Is Lundberg-type inequality for this M/G/∞ driven CL model.

In combination with lower bound, we find following result.

Theorem

In the model with M/G/∞ *driven arrivals,*

$$\lim_{u\to\infty} \frac{1}{u} \log p(u) = -\theta^\star.$$

Shot-noise driven arrivals

Now: CL model in which arrival rate is shot-noise process:

- Let D_i be sequence of i.i.d. non-negative random variables, distributed as e generic random variable D with LST $d(\alpha)$.
- Let $M(t)$ be Poisson process with intensity $\nu > 0$, and let T_i be i-th arrival time it generates.
- Parameter $s > 0$ describes how fast 'shots' decay in time:

$$\Lambda(t) = \sum_{i=1}^{M(t)} D_i e^{-s(t-T_i)}.$$

Main idea behind using shot-noise arrival rate in insurance context: process is well suited to model impact of (randomly arriving) catastrophic events. Floods, windstorms, earthquakes cause a 'pulse' in claim arrival rate, which eventually fades away.

Shot-noise driven arrivals, ctd.

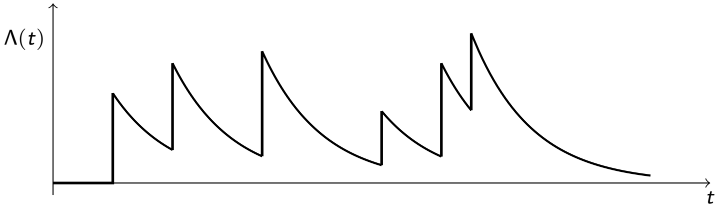

Figure: Arrival rate process $\Lambda(t)$ in shot-noise case.

Shot-noise driven arrivals, ctd.

Assume that (to ensure that $Y(t)$ eventually drifts to $-\infty$)

$$\frac{\mathbb{E}D}{s} \cdot \nu \, \mathbb{E}B < r.$$

Number of claims $N(t)$ in $[0, t]$ is Poisson with random parameter

$$\bar{\Lambda}(t) := \int_0^t \Lambda(u) \, du.$$

To evaluate $\Phi(\alpha)$, above properties lead to

$$\begin{aligned}
\frac{1}{t} \log \mathbb{E} \, e^{-\alpha Y(t)} &= r\alpha + \frac{1}{t} \log \mathbb{E}\left[b(\alpha)^{N(t)} \right] \\
&= r\alpha + \frac{1}{t} \log \mathbb{E} \left[\sum_{i=0}^{\infty} e^{-\bar{\Lambda}(t)} \frac{(\bar{\Lambda}(t))^i}{i!} \big(b(\alpha) \big)^i \right] \\
&= r\alpha + \frac{1}{t} \log \mathbb{E} \, e^{-\bar{\Lambda}(t)(1 - b(\alpha))}.
\end{aligned}$$

Hence to find expression for $\Phi(\alpha)$, we are to compute LST of $\bar{\Lambda}(t)$.

Shot-noise driven arrivals, ctd.

Observe

$$\bar{\Lambda}(t) = \sum_{i=1}^{M(t)} D_i \int_0^{t-T_i} e^{-su}\,du = \sum_{i=1}^{M(t)} D_i \frac{1-e^{-s(t-T_i)}}{s}.$$

Recall: $M(t)$ is Poisson with parameter νt. Also, conditional on number of shot arrivals, each of them arrives at uniformly distributed epoch, independently of each other. Hence,

$$\begin{aligned}\mathbb{E}\, e^{-\alpha\bar{\Lambda}(t)} &= \sum_{k=0}^{\infty} e^{-\nu t}\frac{(\nu t)^k}{k!}\left(\int_0^t \frac{1}{t}\mathbb{E}\exp\left(-\alpha D_i \frac{1-e^{-su}}{s}\right)du\right)^k \\ &= \exp\left(-\nu t + \nu\int_0^t d\left(\alpha\frac{1-e^{-su}}{s}\right)du\right).\end{aligned}$$

Shot-noise driven arrivals, ctd.

Upon combining the above, and sending t to ∞, we have proved the following result.

Proposition

As $t \to \infty$,

$$\frac{1}{t} \log \mathbb{E}\, e^{-\alpha Y(t)} \to \Phi(\alpha) = r\alpha - \nu \left(1 - d\left(\frac{1 - b(\alpha)}{s}\right)\right).$$

Shot-noise driven arrivals, ctd.

Goal: prove that decay rate of $p(u)$ is upper bounded by $-\theta^\star$; use method of Section 2.2.

Starting point: for $u > r$,

$$p(u) \leqslant \mathbb{P}\left(\exists n \in \mathbb{N} : Y(n) \geqslant u - r\right),$$

(use that net cumulative claim process decreases with at most r per unit of time). Hence: upper bound on $p(u)$ that corresponds to *countable* number of events.

Shot-noise driven arrivals, ctd.

Recall definition of $T^\star$, and interpretation of $T^\star u$ as typical time to exceed u.

Intuition behind proof: one term contains contribution of epochs n in order of $T^\star u$ (and is therefore 'dominant'), and term that contains other contributions (and is therefore 'negligible').

Indeed, in combination with union bound,

$$\begin{aligned} p(u) &\leqslant \sum_{n=1}^{T^\star(1+\varepsilon)u} \mathbb{P}\left(Y(n) \geqslant u - r\right) + \sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} \mathbb{P}\left(Y(n) \geqslant u - r\right) \\ &\leqslant \sum_{n=1}^{T^\star(1+\varepsilon)u} \mathbb{P}\left(Y(n) \geqslant u - r\right) + \sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} \mathbb{P}\left(Y(n) \geqslant 0\right), \end{aligned}$$

where $\varepsilon > 0$ will be picked below.

Shot-noise driven arrivals, ctd.

Due to Chernoff bound, for any $\theta > 0$ second term is dominated by

$$\sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} \mathbb{P}\,(Y(n) \geqslant 0) \leqslant \sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} \mathbb{E}\, e^{\theta Y(n)}.$$

Let $\theta^\circ > 0$ be such that $\Phi'(-\theta^\circ) = 0$. (As there is a $\theta^\star$ such that $\Phi(-\theta^\star) = 0$, this θ° exists, and is smaller than $\theta^\star$).

From $\Phi'(0) > 0$ and $\Phi(\alpha)$ being convex, conclude that $\Phi(-\theta^\circ) < 0$. It is readily seen that $-\Phi(-\theta^\circ) = I(0) > 0$; see Exercise 8.1.

Shot-noise driven arrivals, ctd.

Let n be sufficiently large to ensure that

$$\frac{1}{n}\log \mathbb{E}\, e^{\theta^\circ Y(n)} \leqslant \Phi(-\theta^\circ) + \delta = -I(0) + \delta,$$

for some $\delta \in (0, I(0))$; possible due to Proposition (entailing $t^{-1} \log \mathbb{E}\, e^{-\alpha Y(t)} \to \Phi(\alpha)$).

Recognizing geometric sum, we thus find, with $z := \exp(-I(0) + \delta) < 1$,

$$\sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} \mathbb{P}\left(Y(n) \geqslant 0\right) \leqslant \frac{z^{T^\star(1+\varepsilon)u+1}}{1-z}.$$

Shot-noise driven arrivals, ctd.

Now consider first sum (contains most significant contributions, and is therefore dominant). Again using Chernoff bound,

$$\sum_{n=1}^{T^\star(1+\varepsilon)u} \mathbb{P}\left(Y(n) \geqslant u - r\right) \leqslant \sum_{n=1}^{T^\star(1+\varepsilon)u} e^{-\theta^\star(u-r)}\,\mathbb{E}\,e^{\theta^\star Y(n)}$$

$$\leqslant \left(T^\star(1+\varepsilon)u\right) \max_{n=1,\ldots,T^\star(1+\varepsilon)u} e^{-\theta^\star(u-r)}\,\mathbb{E}\,e^{\theta^\star Y(n)}.$$

Then observe that, using that the LST $d(\alpha)$ is decreasing and $1 - b(-\theta^\star) < 0$, for any $t \geqslant 0$,

$$\log \mathbb{E}\,e^{\theta^\star Y(t)} = -r\theta^\star t - \nu t + \nu \int_0^t d\left((1 - b(-\theta^\star))\frac{1 - e^{-su}}{s}\right) du$$

$$\leqslant \left(-r\theta^\star - \nu + \nu d\left(\frac{1 - b(-\theta^\star)}{s}\right)\right) t = \Phi(-\theta^\star)\, t = 0.$$

Shot-noise driven arrivals, ctd.

Combining the above, and using that $u^{-1} \log u \to 0$ as $u \to \infty$, decay rate of first sum is at most $-\theta^\star$:

$$\lim_{u\to\infty} \frac{1}{u} \log \left(\big(T^\star (1+\varepsilon) u \big)\, e^{-\theta^\star (u-r)} \max_{n=1,\dots,T^\star(1+\varepsilon)u} \mathbb{E}\, e^{\theta^\star Y(n)} \right) \leqslant -\theta^\star .$$

Shot-noise driven arrivals, ctd.

We thus have upper bound, with constant ε still to be chosen,

$$\lim_{u\to\infty} \frac{1}{u} \log p(u) \leqslant -\min\left\{\theta^\star, (I(0)-\delta)\, T^\star(1+\varepsilon)\right\}.$$

We pick

$$\varepsilon > \frac{\theta^\star}{T^\star} \frac{1}{I(0)-\delta} - 1 = \frac{I(1/T^\star)}{I(0)-\delta} - 1,$$

where equality follows from $\theta^\star = T^\star\, I(1/T^\star)$; note that number on right-hand side is positive because $I(a)$ is increasing for $a > 0$. Then $\theta^\star < (I(0)-\delta)\, T^\star(1+\varepsilon)$; hence contribution of second sum vanishes. Now recall $\theta^\star$ is lower bound on decay rate as well.

Theorem

In the model with shot-noise driven arrivals,

$$\lim_{u\to\infty} \frac{1}{u} \log p(u) = -\theta^\star.$$

Hawkes driven arrivals

Consider counting process $M(t)$, corresponding to epochs $T_1, T_2, \ldots$ (in that process $M(t)$ increases by 1 at $T_1, T_2, \ldots$), defined as follows. Let, as $\Delta t \downarrow 0$,

$$\mathbb{P}\big(M(t+\Delta t) - M(t) = 1 \,|\, \Lambda(s),\, s \in [0,t]\big) = \Lambda(t)\,\Delta t + o(\Delta t),$$
$$\mathbb{P}\big(M(t+\Delta t) - M(t) = 0 \,|\, \Lambda(s),\, s \in [0,t]\big) = 1 - \Lambda(t)\,\Delta t + o(\Delta t),$$

where, for given parameter $\nu > 0$,

$$\Lambda(t) = \nu + \sum_{i=1}^{M(t)} D_i h(t - T_i) = \nu + \sum_{i:\, T_i \leqslant t} D_i h(t - T_i).$$

Process $\Lambda(t)$ is *Hawkes process.* Function $h(\cdot)$ describes how impact of 'shots' D_i vanishes over time. Goal: find decay rate of $p(u)$ for CL model with Hawkes claim arrivals.

Hawkes driven arrivals, ctd.

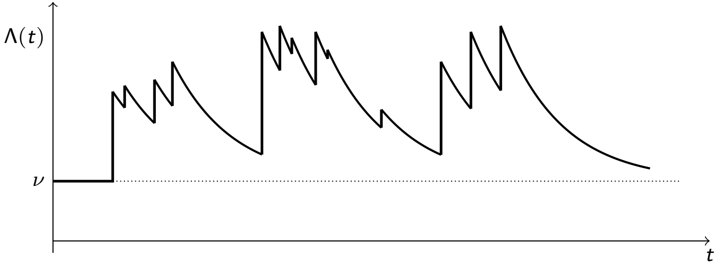

Figure: Arrival rate process $\Lambda(t)$ in Hawkes case.

Hawkes driven arrivals

Current arrival rate depends on observed sequence of past arrival times; ‘self-exciting’.

In insurance context Hawkes arrival rate is used in case one wishes to model effect of claims triggering additional claims.

Require $H\,\mathbb{E}D < 1$, with

$$H := \int_0^\infty h(u)\,du,$$

so that $\Lambda(t)$ does not explode as $t \to \infty$. In addition, require that

$$\frac{1}{1 - H\,\mathbb{E}D} \cdot \nu\,\mathbb{E}B < r,$$

such that $Y(t)$ eventually drifts to $-\infty$.

Hawkes driven arrivals, ctd.

Under above conditions, next result gives (implicit) characterization of limiting Laplace exponent.

Proposition

As $t \to \infty$,

$$\frac{1}{t}\log \mathbb{E}\, e^{-\alpha Y(t)} \to \Phi(\alpha) = r\alpha - \nu\big(1 - \eta(b(\alpha))\big),$$

where $\eta(z)$ is the unique root in $[0,1)$ of fixed-point equation

$$\eta(z) = z\, d((1-\eta(z))H).$$

Hawkes driven arrivals, ctd.

Derivation of this result relies heavily on representation of Hawkes process as *branching process*.

Definition of $\Lambda(t)$ reveals that Hawkes arrival process can be split into

- Poisson process with constant rate ν, in the sequel referred to as *immigrants*,
- arrivals that are induced by the immigrants.

Thus, each of immigrants increases future arrival rate. Arrivals that occur due to this increase, are called *children* of this immigrant. In turn, those children are potentially parents of next generation, and so forth. Useful recursive structure, leads to fixed-point equation for $\eta(z)$.

Hawkes driven arrivals, ctd.

Proof. First objective: analyze transform of $N(t)$ (number of claim arrivals in $[0,t)$).
Let $S(u)$ represent number of children of immigrant, u time units after its own birth, including immigrant itself. Define pgf $\eta(u,z) := \mathbb{E}\, z^{S(u)}$, for $z \leqslant 1$.

Then

$$\mathbb{E}\, z^{N(t)} = \sum_{k=0}^{\infty} e^{-\nu t}\frac{(\nu t)^k}{k!}\left(\frac{1}{t}\int_0^t \eta(u,z)\, du\right)^k$$
$$= \exp\left(-\nu t + \nu\int_0^t \eta(u,z)\, du\right).$$

Hawkes driven arrivals, ctd.

Next task: identification of $\eta(u, z)$, done by studying each cluster separately.

Key element: distributional equality, for fixed $t > 0$ and $u \in [0, t]$,

$$S(u) \stackrel{d}{=} 1 + \sum_{i:T_i \leqslant u} S_i(u - T_i) = 1 + \sum_{i=1}^{K(u)} S_i(u - T_i),$$

where $S_i(u)$ are i.i.d. copies of $S(u)$; here $T_1, T_2, \ldots$ are birth times of corresponding children, and $K(u)$ is inhomogeneous Poisson counting process with rate $Dh(u)$ (conditional on sampled value of D that corresponds to immigrant under consideration, that is).

Interpretation: $S_i(u)$ is number of children of child i (including the child itself).

Hawkes driven arrivals, ctd.

$P_t(s)$: probability that, conditional on a child being born before time t, it was actually already born before s, for $s \leqslant t$. Then,

$$P_t(s) = \frac{\mathbb{P}(K(s) = K(t) = 1)}{\mathbb{P}(K(t) = 1)} = \frac{\mathbb{P}(K(s) = 1, K(t) - K(s) = 0)}{\mathbb{P}(K(t) = 1)}.$$

Conditional on D, we thus find, with

$$r(s,t) := D \int_s^t h(u)\,du, \quad H(t) := \int_0^t h(u)\,du,$$

that

$$P_t(s) = \frac{r(0,s)e^{-r(0,s)} \cdot e^{-r(s,t)}}{r(0,t)e^{-r(0,t)}} = \frac{H(s)}{H(t)};$$

note that D cancels. Now define $p_t(s) := P_t'(s) = h(s)/H(t)$.

Hawkes driven arrivals, ctd.

Appealing to distributional equality, and conditioning on D,

$$\begin{aligned}
\eta(u,z) &= \int_0^\infty \sum_{k=0}^\infty \mathbb{E}\big[z^{S(u)} \,|\, K(u)=k, D=x\big]\,\mathbb{P}(K(u)=k \,|\, D=x)\,\mathbb{P}(D\in dx) \\
&= z\int_0^\infty \sum_{k=0}^\infty \mathbb{E}\left[\prod_{i=1}^k z^{S(u-T_i)}\right] e^{-xH(u)}\frac{(xH(u))^k}{k!}\,\mathbb{P}(D\in dx) \\
&= z\int_0^\infty \sum_{k=0}^\infty \left(\int_0^u \eta(u-s,z)\,p_u(s)\,ds\right)^k e^{-xH(u)}\frac{(xH(u))^k}{k!}\,\mathbb{P}(D\in dx) \\
&= z\int_0^\infty \exp\left(-x\int_0^u (1-\eta(u-s,z))\,h(s)\,ds\right)\mathbb{P}(D\in dx),
\end{aligned}$$

which leads to the fixed-point equation

$$\eta(u,z) = z\,d\left(\int_0^u (1-\eta(u-s,z))\,h(s)\,ds\right).$$

Hawkes driven arrivals, ctd.

Now focus on identifying $\Phi(\alpha)$. First consider $\mathbb{E}\, e^{-\alpha Y(t)}$, which we express in terms of $\eta(u, b(\alpha))$. Observe:

$$\frac{1}{t}\log \mathbb{E}\, e^{-\alpha Y(t)} = r\alpha - \frac{1}{t}\log \mathbb{E}\big[b(\alpha)^{N(t)}\big].$$

Hence,

$$\begin{aligned}\Phi(\alpha) &= r\alpha - \nu\left(1 - \lim_{t\to\infty}\frac{1}{t}\int_0^t \eta(u, b(\alpha))\, du\right)\\ &= r\alpha - \nu\left(1 - \eta(\infty, b(\alpha))\right),\end{aligned}$$

where, because of fixed-point equation for $\eta(u, z)$, it follows that $\eta(\infty, z) = \eta(z)$ solves fixed-point equation featuring in statement of Proposition.

Hawkes driven arrivals, ctd.

Theorem

In the model with Hawkes driven arrivals,

$$\lim_{u\to\infty}\frac{1}{u}\log p(u) = -\theta^\star.$$

Hawkes driven arrivals, ctd.

Proof. Completely analogous to that of case with shot-noise driven arrivals, except that we have to find a new proof of $\log \mathbb{E}\, e^{\theta^\star Y(t)} \leqslant 0$.

From definition of $S(u)$ it follows that $S(\infty) \geqslant S(u)$ for all $u \geqslant 0$, so that for all $z \in [0, 1]$ we have that $\eta(u, z) \leqslant \eta(\infty, z)$. Using $b(-\theta^\star) < 1$,

$$\int_0^t \eta(u, b(-\theta^\star))\, du \leqslant \int_0^t \eta(\infty, b(-\theta^\star))\, du = t\, \eta(\infty, b(-\theta^\star)).$$

Hence,

$$\begin{aligned} \log \mathbb{E}\, e^{\theta^\star Y(t)} &= -r\theta^\star t - \nu t + \nu \int_0^t \eta(u, b(-\theta^\star))\, du \\ &\leqslant \Big(-r\theta^\star - \nu\big(1 - \eta(b(-\theta^\star))\big)\Big)\, t = \Phi(-\theta^\star)\, t = 0. \end{aligned}$$

CHAPTER IX: DEPENDENCE BETWEEN CLAIM SIZES AND INTERARRIVAL TIMES

Dependence between claim sizes and interarrival times: main ideas

This chapter: dependence between claims and interarrival times.

- Claim size being correlated with previous interarrival time;
- interarrival time being correlated with previous claim size.

Objective: determine transform of time-dependent ruin probability.

Claim size correlated with previous interarrival time

Model 1. Claim size directly determines parameter of exponential distribution of preceding interclaim time.

Concretely: if claim size is $v > 0$, then length of interval between previous claim's arrival time and this claim's arrival time has exponential distribution with parameter $\lambda(v) > 0$.

The time-dependent ruin probability $p(u, t)$ and the double transform $\pi(\alpha, \beta)$ are defined in the usual manner.

Claim size correlated with previous interarrival time, ctd.

Approach of Section 1.3: $\pi(\alpha,\beta)$ is written as sum of $\pi_1(\alpha,\beta)$ (ruin due to first arriving claim) and $\pi_2(\alpha,\beta)$ (ruin occurring later).

First contribution:

$$\pi_1(\alpha,\beta) = \int_0^\infty \frac{\lambda(v)}{\lambda(v)+\beta}\left(\frac{1-e^{-\alpha v}}{\alpha} - \frac{e^{-(\lambda(v)+\beta)v/r} - e^{-\alpha v}}{\alpha - (\lambda(v)+\beta)/r}\right)\mathbb{P}(B \in dv).$$

With $s(v,\beta)$ defined as $(\lambda(v)+\beta)/r$, this quantity can be interpreted as

$$\pi_1(\alpha,\beta) = \mathbb{E}\left(\frac{\lambda(B)}{\lambda(B)+\beta}\left(\frac{1-e^{-\alpha B}}{\alpha} - \frac{e^{-s(B,\beta)\,B} - e^{-\alpha B}}{\alpha - s(B,\beta)}\right)\right),$$

which we can calculate (as we know the distribution of B).

Claim size correlated with previous interarrival time, ctd.

Second contribution, as in Section 1.3:

$$\pi_2(\alpha,\beta) = \int_0^\infty \frac{\lambda(v)}{r}\left(\int_v^\infty \frac{e^{-s(v,\beta)w} - e^{-\alpha w}}{\alpha - s(v,\beta)} p(w-v,\, T_\beta)\, dw\right)\mathbb{P}(B\in dv).$$

Directly seen: $\pi_2(\alpha,\beta)$ can be written as difference of

$$\begin{aligned}
\pi_2^+(\alpha,\beta) &:= \int_0^\infty \frac{\lambda(v)}{r}\left(\int_v^\infty \frac{e^{-s(v,\beta)w}}{\alpha - s(v,\beta)} p(w-v,\, T_\beta)\, dw\right)\mathbb{P}(B\in dv)\\
&= \int_0^\infty \frac{\lambda(v)}{r(\alpha - s(v,\beta))} e^{-s(v,\beta)v}\left(\int_0^\infty e^{-s(v,\beta)w}\, p(w,\, T_\beta)\, dw\right)\mathbb{P}(B\in dv)\\
&= \mathbb{E}\left(\frac{\lambda(B)}{r(\alpha - s(B,\beta))}\, e^{-s(B,\beta)\,B}\, \pi\big(s(B,\beta),\beta\big)\right),
\end{aligned}$$

which is an expression that we cannot further evaluate (yet), and

Claim size correlated with previous interarrival time, ctd.

$$\begin{aligned}
\pi_2^-(\alpha,\beta) &:= \int_0^\infty \frac{\lambda(v)}{r}\left(\int_v^\infty \frac{e^{-\alpha w}}{\alpha - s(v,\beta)} p(w-v,T_\beta)\,dw\right)\mathbb{P}(B\in dv)\\
&= \int_0^\infty \frac{\lambda(v)}{r(\alpha - s(v,\beta))} e^{-\alpha v}\mathbb{P}(B\in dv)\int_0^\infty e^{-\alpha w}\,p(w,T_\beta)\,dw\\
&= \mathbb{E}\left(\frac{\lambda(B)}{r(\alpha - s(B,\beta))}e^{-\alpha B}\right)\,\pi(\alpha,\beta).
\end{aligned}$$

Observe that

$$\pi^\circ(\alpha,\beta) := \mathbb{E}\left(\frac{\lambda(B)}{r(\alpha - s(B,\beta))}\,e^{-\alpha B}\right) = -\mathbb{E}\left(\frac{\lambda(B)}{\lambda(B)+\beta - r\alpha}\,e^{-\alpha B}\right),$$

which we can evaluate, as we know distribution of B.

Claim size correlated with previous interarrival time, ctd.

Isolate the quantity of our interest:

$$\pi(\alpha,\beta) = \frac{\pi_1(\alpha,\beta) + \pi_2^+(\alpha,\beta)}{1+\pi^\circ(\alpha,\beta)}.$$

But: $\pi_2^+(\alpha,\beta)$ is not known yet.

Consider case that claim size distribution is given by

$$\mathbb{P}(B \leqslant v) = \sum_{i=1}^{d} p_i U(v - b_i),$$

with $U(\cdot)$ unit step function and $p_1,\ldots,p_d > 0$, $\sum_{i=1}^d p_i = 1$. Hence: there are K possible claim arrival rates $\lambda(b_1),\ldots,\lambda(b_d)$, assuming (wlog) that $\lambda(b_1) \leqslant \lambda(b_2) \leqslant \ldots \leqslant \lambda(b_d)$.

Claim size correlated with previous interarrival time, ctd.

Then

$$1+\pi^\circ(\alpha,\beta) = 1-\sum_{i=1}^{d} p_i \frac{\lambda(b_i)}{\lambda(b_i)+\beta-r\alpha} e^{-\alpha b_i} = \frac{f(\alpha)-g(\alpha)}{f(\alpha)},$$

where

$$f(\alpha) := \prod_{i=1}^{d} (\lambda(b_i)+\beta-r\alpha),$$

and

$$g(\alpha) := \sum_{i=1}^{d} p_i \frac{\lambda(b_i)}{\lambda(b_i)+\beta-r\alpha} e^{-\alpha b_i} f(\alpha).$$

Claim size correlated with previous interarrival time, ctd.

Apply 'Rouché' to $f(\alpha) - g(\alpha)$: it has exactly d zeroes in right-half plane.

Inspection of behavior of $1 + \pi^{\circ}(\alpha, \beta)$ at the asymptotes $\alpha = s(b_i, \beta)$, $i = 1, \ldots, d$: these d zeroes of $1 + \pi^{\circ}(\alpha, \beta)$ (say, $\alpha_1^{\star}(\beta), \ldots, \alpha_d^{\star}(\beta)$) are all real, exactly one being located in $(0, s(b_1, \beta))$, one in $(s(b_1, \beta), s(b_2, \beta))$, etc.

For those zeroes, numerator (i.e., $\pi_1(\alpha, \beta) + \pi_2^{+}(\alpha, \beta)$) should be zero, too. Leads to d linear equations in the d remaining unknowns $\pi(s(b_j, \beta), \beta)$ featuring in $\pi_2^{+}(\alpha, \beta)$.

Thus, $\pi(\alpha, \beta)$ is completely determined.

Claim size correlated with previous interarrival time, ctd.

Model 2. Sequence $B_1, B_2, \ldots$ represents the i.i.d. claim sizes. $V_1, V_2, \ldots$ is second sequence of i.i.d. random variables, independent of the claim sizes.

After n-th claim arrival, new claim interarrival time A_{n+1}, threshold value V_{n+1} and claim size B_{n+1} are drawn. If $B_{n+1} = v$ and $z := v/V_{n+1}$, then A_{n+1} is exponentially distributed with parameter $\lambda(z) > 0$. We consider the case that $\lambda(z)$ attains values in $[0, D]$ for some $D > 0$.

Claim size correlated with previous interarrival time, ctd.

Objective: time-dependent ruin probability $p(u, T_\beta)$.

As $\Delta t \downarrow 0$,

$$p(u, T_\beta) =$$

$$\left(1 - \int_0^D \lambda(z)\,\Delta t \int_0^\infty \mathbb{P}(B \in dv)\, d_z\mathbb{P}(V < v/z) - \beta\,\Delta t\right) p(u + r\,\Delta t, T_\beta)$$

$$+ \int_0^D \lambda(z)\,\Delta t \int_u^\infty \mathbb{P}(B \in dv)\, d_z\mathbb{P}(V < v/z)$$

$$+ \int_0^D \lambda(z)\,\Delta t \int_0^u \mathbb{P}(B \in dv)\, d_z\mathbb{P}(V < v/z)\, p(u - v, T_\beta),$$

up to $o(\Delta t)$ terms.

Claim size correlated with previous interarrival time, ctd.

Define

$$\chi(\alpha) := \int_0^D \lambda(z) \int_0^\infty e^{-\alpha v}\mathbb{P}(B \in dv)\, d_z\mathbb{P}(V < v/z).$$

Follow standard procedure: subtract $p(u + r\,\Delta t, T_\beta)$ from both sides, divide by Δt, and take limit $\Delta \downarrow 0$. We obtain

$$\begin{aligned}
-r\frac{\partial}{\partial u}p(u, T_\beta) = -\big(\chi(0) + \beta\big)p(u, T_\beta) + \\
\int_0^D \lambda(z) \int_u^\infty \mathbb{P}(B \in dv)\, d_z\mathbb{P}(V < v/z) + \\
\int_0^D \lambda(z) \int_0^u \mathbb{P}(B \in dv)\, d_z\mathbb{P}(V < v/z)\, p(u - v, T_\beta).
\end{aligned}$$

Next step: transform with respect to u, i.e., multiply both sides by $e^{-\alpha u}$ and integrate over u.

Claim size correlated with previous interarrival time, ctd.

Define $\pi(\alpha,\beta)$ in usual manner, and denote $f(\beta) := p(0+, T_\beta)$.

Proposition

For any $\alpha, \beta > 0$,

$$-r\alpha\,\pi(\alpha,\beta) + r\,f(\beta)$$
$$= -\big(\chi(0)+\beta\big)\,\pi(\alpha,\beta) + \frac{\chi(0)-\chi(\alpha)}{\alpha} + \chi(\alpha)\,\pi(\alpha,\beta).$$

Claim size correlated with previous interarrival time, ctd.

Next goal: identify $\pi(\alpha,\beta)$, which requires $f(\beta)$. Observe that

$$\pi(\alpha,\beta) = \frac{rf(\beta) - (\chi(0)-\chi(\alpha))/\alpha}{r\alpha - \chi(0) + \chi(\alpha) - \beta}.$$

Notice: $\chi(\alpha)$ is Laplace transform of probability distribution, and hence convex and decreasing. Therefore, denominator has exactly one positive real zero $\alpha^\star(\beta)$ for every $\beta > 0$.

For any $\beta > 0$, root of denominator is also root of numerator, so that

$$f(\beta) = \frac{1}{r}\frac{\chi(0)-\chi(\alpha^\star(\beta))}{\alpha^\star(\beta)}.$$

Theorem

For any $\alpha,\beta > 0$,

$$\pi(\alpha,\beta) = \frac{(\chi(0)-\chi(\alpha^\star(\beta)))/\alpha^\star(\beta) - (\chi(0)-\chi(\alpha))/\alpha}{r\alpha - \chi(0) + \chi(\alpha) - \beta}.$$

Interarrival time being correlated with previous claim size

Mechanism is similar to Model 2 discussed above. Consider $\boldsymbol{z} \equiv (z_0, \ldots, z_d)$ such that $0 = z_0 < z_1 < \cdots < z_d = \infty$. Claim sizes $B_1, B_2, \ldots$ are i.i.d. (distributed as B). In addition, $V_1, V_2, \ldots$ are i.i.d., independent of claim sizes (distributed as V).

If claim B_n is in $[z_{i-1}V_n, z_i V_n)$, then time until next claim is exponentially distributed with rate $\lambda_i > 0$.

Key object of interest: for $i = 1, \ldots, d$,

$$p_i(u,t) := \mathbb{P}(\exists s \in [0,t] : X_u(s) \leqslant 0 \,|\, J(0) = i);$$

$\{J(0) = i\}$ corresponds to scenario that arrival rate at time 0 is λ_i.

Objective: characterize $p_i(u,t)$ through its double transform

$$\pi_i(\alpha,\beta) = \int_0^\infty \int_0^\infty \beta e^{-\alpha u - \beta t} p_i(u,t)\, du\, dt.$$

Interarrival time correlated with previous claim size, ctd.

By familiar method, up to $o(\Delta t)$ terms, with T_β exponentially distributed with parameter β, as $\Delta t \downarrow 0$,

$$p_i(u, T_\beta) = \lambda_i \, \Delta t \sum_{j=1}^{d} \int_0^u \mathbb{P}(B \in dv) \, \mathbb{P}\Big(V \in [\frac{v}{z_j}, \frac{v}{z_{j-1}})\Big) \, p_j(u - v, T_\beta)$$
$$+ \; \lambda_i \, \Delta t \, \mathbb{P}(B \geqslant u) + \big(1 - \lambda_i \, \Delta t - \beta \, \Delta t\big) \, p_i(u + r \, \Delta t, T_\beta).$$

Subtracting $p_i(u + r \, \Delta t, T_\beta)$ from both sides and dividing full equation by Δt, sending Δt to 0:

$$-r \frac{\partial}{\partial u} p_i(u, T_\beta) = \lambda_i \sum_{j=1}^{d} \int_0^u \mathbb{P}(B \in dv) \, \mathbb{P}\Big(V \in [\frac{v}{z_j}, \frac{v}{z_{j-1}})\Big) \, p_j(u - v, T_\beta) +$$
$$\lambda_i \, \mathbb{P}(B \geqslant u) - (\lambda_i + \beta) \, p_i(u, T_\beta).$$

Interarrival time correlated with previous claim size, ctd.

Define, for $j = 1, \ldots, d$,

$$\chi_j(\alpha) := \int_0^\infty e^{-\alpha v}\, \mathbb{P}(B \in dv)\, \mathbb{P}\Big(V \in [\frac{v}{z_j}, \frac{v}{z_{j-1}})\Big).$$

Multiply equation by $e^{-\alpha u}$ and integrate over u. Integrating by parts, and recognizing convolution in right-hand side, and denoting $f_i(\beta) := p_i(0+, T_\beta)$, we obtain following result.

Proposition

For any $\alpha, \beta > 0$, and $i = 1, \ldots, d$,

$$-r\alpha\, \pi_i(\alpha, \beta) + rf_i(\beta) = \lambda_i \sum_{j=1}^{d} \chi_j(\alpha)\, \pi_j(\alpha, \beta) + \lambda_i \frac{1 - b(\alpha)}{\alpha} - (\lambda_i + \beta)\, \pi_i(\alpha, \beta).$$

Interarrival time correlated with previous claim size, ctd.

Proposition provides equations, containing d unknowns $f_1(\beta),\dots,f_d(\beta)$. Observe that these equations yield that, for any pair $i,j\in\{1,\dots,d\}$,

$$\frac{\big(r\alpha-\lambda_i-\beta\big)\pi_i(\alpha,\beta)-rf_i(\beta)}{\lambda_i}=\frac{\big(r\alpha-\lambda_j-\beta\big)\pi_j(\alpha,\beta)-rf_j(\beta)}{\lambda_j},$$

or, equivalently,

$$\pi_j(\alpha,\beta)=A_{ij}(\alpha,\beta)\,\pi_i(\alpha,\beta)+B_{ij}(\alpha,\beta),$$

where

$$A_{ij}(\alpha,\beta):=\frac{\lambda_j}{\lambda_i}\frac{r\alpha-\lambda_i-\beta}{r\alpha-\lambda_j-\beta},B_{ij}(\alpha,\beta):=-\frac{\lambda_j}{\lambda_i}\frac{rf_i(\beta)}{r\alpha-\lambda_j-\beta}+\frac{rf_j(\beta)}{r\alpha-\lambda_j-\beta}.$$

Hence,

$$\begin{aligned}&(-r\alpha+\lambda_i+\beta\,)\,\pi_i(\alpha,\beta)+rf_i(\beta)\\&\qquad=\lambda_i\sum_{j=1}^{d}\chi_j(\alpha)\,\big(A_{ij}(\alpha,\beta)\,\pi_i(\alpha,\beta)+B_{ij}(\alpha,\beta)\big)+\lambda_i\frac{1-b(\alpha)}{\alpha}.\end{aligned}$$

Interarrival time correlated with previous claim size, ctd.

Hence, $\pi_i(\alpha,\beta)$ can be solved:

$$\pi_i(\alpha,\beta) = \frac{rf_i(\beta) - \lambda_i \sum_{j=1}^{d} \chi_j(\alpha)\, B_{ij}(\alpha,\beta) - \lambda_i(1-b(\alpha))/\alpha}{r\alpha - \lambda_i + \lambda_i \sum_{j=1}^{d} \chi_j(\alpha)\, A_{ij}(\alpha,\beta) - \beta}.$$

Any zero of denominator should be zero of numerator as well. Minor computation (for $\beta > 0$ given and for $i = 1,\ldots,d$): solve $H(\alpha) = 1$, with

$$H(\alpha) = \sum_{j=1}^{d} \frac{\lambda_j}{\lambda_j + \beta - r\alpha} \chi_j(\alpha).$$

Interarrival time correlated with previous claim size, ctd.

Observe: $H(0) < 1$, whereas $H(\alpha)$ approaches 0 from below as $\alpha \to \infty$. Assume (wlog) $\lambda_1 < \cdots < \lambda_d$. With $\alpha_0 = 0$ and $\alpha_j := (\lambda_j + \beta)/r$, we have that

$$\lim_{\alpha \uparrow \alpha_j} H(\alpha) = \infty, \quad \lim_{\alpha \downarrow \alpha_j} H(\alpha) = -\infty,$$

for $j = 1, \ldots, d$.

Hence, for all $\beta > 0$, there is solution to $H(\alpha) = 1$ in each of intervals (α_{j-1}, α_j), for $j = 1, \ldots, d$. We call these zeroes $\alpha_1^\star(\beta), \ldots, \alpha_d^\star(\beta)$, which are necessarily zeroes of numerator as well.

Interarrival time correlated with previous claim size, ctd.

Consequently, for $\alpha = \alpha_1^\star(\beta), \ldots, \alpha_d^\star(\beta)$,

$$0 = rf_i(\beta) - \lambda_i \sum_{j=1}^{d} \chi_j(\alpha)\, B_{ij}(\alpha, \beta) - \lambda_i \frac{1 - b(\alpha)}{\alpha}$$

$$= rf_i(\beta) \left(1 - H(\alpha) \right) + \lambda_i r \sum_{j=1}^{d} \frac{\chi_j(\alpha)}{\lambda_j + \beta - r\alpha} f_j(\beta) - \lambda_i \frac{1 - b(\alpha)}{\alpha} =: C_i(\alpha, \beta).$$

Using that $H(\alpha_j^\star(\beta)) = 1$ for $j = 1, \ldots, d$, after dividing by λ_i:

$$r \sum_{j=1}^{d} \frac{\chi_j(\alpha_j^\star(\beta))}{\lambda_j + \beta - r\alpha_j^\star(\beta)} f_j(\beta) = \frac{1 - b(\alpha_j^\star(\beta))}{\alpha_j^\star(\beta)},$$

for $j = 1, \ldots, d$.

Observe: d linear equations do not depend on i anymore, so that $f_j(\beta)$ can be identified.

Interarrival time correlated with previous claim size, ctd.

Theorem

For any $\alpha \geqslant 0$ and $\beta > 0$, and $i = 1, \ldots, d$,

$$\pi_i(\alpha, \beta) = \frac{C_i(\alpha, \beta)}{r\alpha - \lambda_i + \lambda_i \sum_{j=1}^{d} \chi_j(\alpha)\, A_{ij}(\alpha, \beta) - \beta},$$

where the $f_j(\beta)$, for $j = 1, \ldots, d$, follow from the d linear equations.

A more general Markov-dependent risk model

Let A_i denote time between the arrival of $(i-1)$-st and i-th claim and $A_0 = B_0 = 0$. Then

$$\begin{aligned}\mathbb{P}(A_{n+1} \leqslant x, B_{n+1} \leqslant y, Z_{n+1} = j \mid Z_n = i, (A_m, B_m, Z_m), m \in \{0, 1, \ldots, n\}) \\ = \mathbb{P}(A_1 \leqslant x, B_1 \leqslant y, Z_1 = j \mid Z_0 = i) = (1 - e^{-\lambda_i x}) p_{ij} \, F_j(y),\end{aligned}$$

where $(Z_n)_{n \in \mathbb{N}}$ is irreducible discrete-time Markov chain with finite state space $\{1, \ldots, d\}$ and transition matrix P consisting of transition probabilities $p_{ij} := \mathbb{P}(Z_{n+1} = j \mid Z_n = i)$.

Thus: at claim arrival, Markov chain jumps to state j, and distribution function $F_j(\cdot)$ of the claim size depends on new state j. Then next interarrival time is exponentially distributed with parameter λ_j.

A more general Markov-dependent risk model, ctd.

By and large, same strategy can be followed as before: set up differential equation for $p_i(u, T_\beta)$, transform with respect to u.

Leads to expressions for $\pi_i(\alpha, \beta)$, in terms of d unknowns. Identification of these unknowns is a bit more involved, though (requires some complex analysis).

CHAPTER X: ADVANCED BANKRUPTCY CONCEPTS

Advanced bankruptcy concepts: main ideas

This chapter: CL model but with a focus on *bankruptcy* rather than ruin.

Three different bankruptcy criteria are studied:

- Reserve level process drops below 0 at Poisson inspection;
- time in first excursion (of reserve level process) below 0 exceeds threshold;
- total time (of reserve level process) below 0 exceeds threshold.

Objective: determine transform of bankruptcy probability.

Poisson inspection

Surplus level is only observed at Poissonian inspection epochs $S_1, S_2, \ldots$, i.e., *not* continuously in time.

Times between two subsequent inspections (i.e., $S_n - S_{n-1}$ for $n \in \mathbb{N}$, with $S_0 \equiv 0$) are i.i.d. exponentially distributed random variables, say with parameter $\omega > 0$.

Quantity of interest: (time-dependent) *bankruptcy probability*

$$\bar{p}(u,t) := \mathbb{P}(\exists n \in \mathbb{N} : S_n \leqslant t, X_u(S_n) \leqslant 0).$$

Clearly, $\bar{p}(u,t) \leqslant p(u,t)$.

Poisson inspection, ctd.

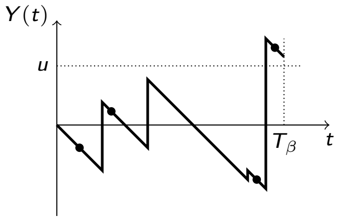

Figure: Scenario with ruin and bankruptcy (left panel), and scenario with ruin but no bankruptcy (right panel). Black dots indicate Poisson inspection epochs.

Poisson inspection, ctd.

Increments of $Y(t)$ between two consecutive inspections, i.e.,

$$Z_n := Y(S_n) - Y(S_{n-1}),$$

which form sequence of i.i.d. random variables, distributed as generic random variable Z.

Killing time T_β is exponentially distributed with parameter β, independent of surplus process.

Number of observations before killing, denoted by $N \equiv N_{\beta,\omega}$, has geometric distribution with success probability $\beta/(\beta+\omega)$:

$$\mathbb{P}(N = n) = \left(\frac{\omega}{\beta+\omega}\right)^n \frac{\beta}{\beta+\omega}, \quad n = 0, 1, \ldots .$$

(Check!)

Poisson inspection, ctd.

Define associated running maximum process:

$$\bar{Y}_{\beta,\omega} := \sup_{n=0,1,\ldots,N_{\beta,\omega}} Y(S_n) = \sup_{n=0,1,\ldots,N_{\beta,\omega}} \sum_{m=1}^{n} Z_m;$$

maximum over empty set is zero.

Notice that

$$\bar{p}(u, T_\beta) = \mathbb{P}(\bar{Y}_{\beta,\omega} \geqslant u).$$

Goal: analyze $\bar{p}(u, T_\beta)$ by evaluating $\mathbb{P}(\bar{Y}_{\beta,\omega} \geqslant u)$. Important role is played by transient waiting time in M/G/1 queue.

Poisson inspection, ctd.

Focus on transform of waiting time W_N of the N-th client in M/G/1 (starting empty), with N geometrically distributed with success probability $q \in [0, 1]$.

Arrival rate is $\nu > 0$. Jobs given by the sequence of i.i.d. random variables $(D_n)_{n\in\mathbb{N}}$, distributed as generic random variable D with LST $\delta(\alpha) = \mathbb{E}\, e^{-\alpha D}$.

Poisson inspection, ctd.

Lindley recursion: W_{n+1} can be expressed in terms of W_n through,

$$W_{n+1} = \max\{W_n + D_n - E_{n+1}, 0\},$$

with $W_0 = 0$ (suppressed elsewhere) and $(E_n)_{n\in\mathbb{N}}$ exponentially distributed with parameter ν.

This leads to identity, with $w_n(\alpha) := \mathbb{E}\, e^{-\alpha W_n}$,

$$w_{n+1}(\alpha) = \int_0^\infty \int_0^\infty e^{-\alpha \max\{x-y,0\}}\, \nu e^{-\nu y}\, dy\, \mathbb{P}(W_n + D_n \in dx).$$

Distinguishing between the cases $x \leqslant y$ and $x > y$, this expression equals

$$\frac{\nu}{\alpha-\nu}\int_0^\infty (e^{-\nu x} - e^{-\alpha x})\, \mathbb{P}(W_n + D_n \in dx) + \int_0^\infty e^{-\nu x}\, \mathbb{P}(W_n + D_n \in dx),$$

which, using that W_n and D_n are independent, leads to

$$w_{n+1}(\alpha) = \frac{\alpha\, w_n(\nu)\, \delta(\nu) - \nu\, w_n(\alpha)\, \delta(\alpha)}{\alpha - \nu}.$$

Poisson inspection, ctd.

Now: find an expression of waiting time of the N-th client.

Multiplying both sides by $(1-q)^n q$ and summing over $n = 0, 1, \ldots$:

$$\mathbb{E}\, e^{-\alpha W_N} = \frac{q(\alpha - \nu) + \alpha(1-q)\,\delta(\nu)\,\mathbb{E}\, e^{-\nu W_N}}{\alpha - \nu + \nu(1-q)\,\delta(\alpha)}.$$

Constant $\mathbb{E}\, e^{-\nu W_N}$ can be identified in the usual manner: there is (unique) $\alpha_0 \in (0, \nu)$ such that the denominator vanishes, so that numerator should be equal to 0 for this α_0.

Poisson inspection, ctd.

This zero α_0 can be rewritten in a convenient form. With

$$\Phi(\alpha) := \mathbb{E}\, e^{-\alpha(D_n - E_{n+1})} = \frac{\nu}{\nu - \alpha}\delta(\alpha),$$

we are to solve $\Phi(\alpha_0) = 1/(1-q)$. Hence, defining $\Psi(\cdot)$ as the (right-)inverse of $\Phi(\cdot)$,

$$\alpha_0 = \Psi\left(\frac{1}{1-q}\right).$$

There is exactly one real root between 0 and ν. (Check!)

Hence,

$$\mathbb{E}\, e^{-\nu W_N} = \frac{q}{1-q}\,\frac{\nu - \alpha_0}{\alpha_0}\,\frac{1}{\delta(\nu)}.$$

Poisson inspection, ctd.

We found following result, which is counterpart of Theorem 1.1.

Lemma

For $\alpha > 0$ and $q \in [0,1]$,

$$\begin{aligned}\mathbb{E}\, e^{-\alpha W_N} &= q\,\frac{\alpha-\nu+(\nu-\alpha_0)\,\alpha/\alpha_0}{\alpha-\nu+\nu(1-q)\,\delta(\alpha)}\\ &= \left(\frac{\alpha}{\alpha_0}-1\right)\,\frac{q\nu}{\alpha-\nu+\nu(1-q)\,\delta(\alpha)}.\end{aligned}$$

Poisson inspection, ctd.

Now: relate waiting times to associated running maximum process.

Lemma

Denote $F_n := D_{n-1} - E_n$. *For any* $n = 0, 1, \ldots,$

$$W_n \stackrel{d}{=} \max_{m=0,1,\ldots,n} \sum_{i=1}^{m} F_i =: G_n.$$

Proof. By iterating the Lindley recursion,

$$\begin{aligned} W_n &= \max\{W_{n-1} + F_n, 0\} = \max\{\max\{W_{n-2} + F_{n-1}, 0\} + F_n, 0\} \\ &= \max\{W_{n-2} + F_{n-1} + F_n, F_n, 0\}. \end{aligned}$$

After n iterations:

$$W_n = \max\left\{\max_{m=1,\ldots,n} \sum_{i=m}^{n} F_i, 0\right\}.$$

Stated follows by reversing time.

Poisson inspection, ctd.

By combining lemmas: expression for transform of running maximum process $(G_n)_{n\in\mathbb{N}}$, with number of terms having a geometric distribution.

Conclude that

$$\begin{aligned}\mathbb{E}\, e^{-\alpha G_N} &= q\,\frac{\alpha-\nu+(\nu-\alpha_0)\,\alpha/\alpha_0}{\alpha-\nu+\nu(1-q)\,\delta(\alpha)}\\ &= \left(\frac{\alpha}{\alpha_0}-1\right)\,\frac{q\nu}{\alpha-\nu+\nu(1-q)\,\delta(\alpha)}.\end{aligned}$$

Poisson inspection, ctd.

Now, for $\alpha \geqslant 0$ and $\beta > 0$, focus on

$$\bar{\varrho}(\alpha, \beta) := \mathbb{E}\, e^{-\alpha \bar{Y}_{\beta,\omega}}.$$

Recall: $\bar{Y}_{\beta,\omega}$ is running maximum of partial sums of $(Z_n)_{n\in\mathbb{N}}$, over maximally $N_{\beta,\omega}$ terms.

By 'Wiener-Hopf' (Proposition 1.2), we can decompose increments as

$$Z = Z^+ - Z^-,$$

with Z^+ and Z^- both non-negative and independent.

Next step: consider Z^- and Z^+ in greater detail.

Poisson inspection, ctd.

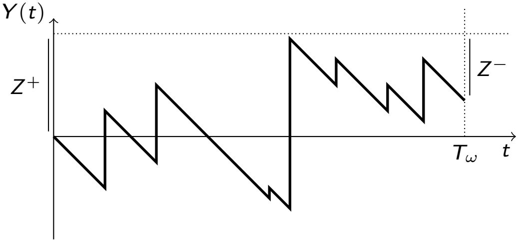

Figure: The decomposition $Z = Z^+ - Z^-$.

Poisson inspection, ctd.

- Observe: Z^- is distributed, again by 'Wiener-Hopf', as running minimum of $Y(t)$ over a period that is exponentially distributed with parameter $\beta + \omega$. (Why?)
  Section 1.3: this running minimum has exponential distribution with parameter $\psi(\beta + \omega)$. Recall: $\psi(\cdot)$ is right inverse of $\varphi(\alpha) = r\alpha - \lambda(1 - b(\alpha))$.
- Results of Section 1.3:

$$\mathbb{E}\, e^{-\alpha Z^+} = \frac{\alpha - \psi(\beta + \omega)}{\varphi(\alpha) - \beta - \omega} \frac{\beta + \omega}{\psi(\beta + \omega)}.$$

Poisson inspection, ctd.

Consequence of above observations: $\bar{Y}_{\beta,\omega}$ can be interpreted as waiting time of $N_{\beta,\omega}$-th client in M/G/1 queue (with service speed 1), where $N_{\beta,\omega}$ is geometrically distributed with success probability $q := \beta/(\beta+\omega)$, arrival rate is $\nu := \psi(\beta+\omega)$, and jump sizes D are distributed as Z_+, i.e.,

$$\delta(\alpha) := \frac{\alpha - \psi(\beta+\omega)}{\varphi(\alpha) - \beta - \omega} \frac{\beta+\omega}{\psi(\beta+\omega)}.$$

Poisson inspection, ctd.

Combining the above, $\bar{\varrho}(\alpha,\beta)$ equals

$$\frac{\beta}{\beta+\omega}\cdot\frac{\alpha-\psi(\beta+\omega)+(\psi(\beta+\omega)-\alpha_0)\dfrac{\alpha}{\alpha_0}}{\alpha-\psi(\beta+\omega)+\psi(\beta+\omega)\dfrac{\omega}{\beta+\omega}\,\dfrac{\alpha-\psi(\beta+\omega)}{\varphi(\alpha)-\beta-\omega}\,\dfrac{\beta+\omega}{\psi(\beta+\omega)}}.$$

Elementary calculus: this expression can be simplified to

$$\frac{\beta}{\beta+\omega}\cdot\frac{\psi(\beta+\omega)}{\alpha-\psi(\beta+\omega)}\left(\frac{\alpha}{\alpha_0}-1\right)\frac{\varphi(\alpha)-\beta-\omega}{\varphi(\alpha)-\beta}.$$

Poisson inspection, ctd.

Next step: identify α_0. Note: $\alpha = \psi(\beta + \omega)$ is root of denominator, but automatically of numerator as well.

Therefore: consider other root of numerator, i.e., $\alpha_0 = \psi(\beta)$. Rearranging the factors in numerators and denominators, we find following result.

Theorem

For any $\alpha \geqslant 0$ and $\beta > 0$,

$$\bar{\varrho}(\alpha, \beta) = \frac{\alpha - \psi(\beta)}{\varphi(\alpha) - \beta} \frac{\beta}{\psi(\beta)} \frac{\varphi(\alpha) - \beta - \omega}{\alpha - \psi(\beta + \omega)} \frac{\psi(\beta + \omega)}{\beta + \omega}.$$

Poisson inspection, ctd.

This theorem: generalization of Theorem 1.1. Indeed, as $\omega \to \infty$, which corresponds to 'permanent inspection', we recover Theorem 1.1.

In addition (Check!),

$$\bar{\varrho}(\alpha, \beta) = \frac{\varrho(\alpha, \beta)}{\varrho(\alpha, \beta + \omega)}.$$

Following remarkable distributional equality is obtained.

Theorem

For any $\beta, \omega > 0$,

$$\bar{Y}(T_\beta) \stackrel{d}{=} \bar{Y}(T_{\beta+\omega}) + \bar{Y}_{\beta,\omega},$$

with random variables on right-hand side independently sampled.

Poisson inspection, ctd.

Now consider information loss due to Poisson inspection. Section 2.2: approximation for $p(u)$ for u large in case of light-tailed input.

We found $\gamma, \theta^\star > 0$ such that, as $u \to \infty$,

$$p(u)\, e^{\theta^\star u} \to \gamma.$$

Question: how much lower is $\bar{p}(u) := \bar{p}(u, \infty)$ than $p(u)$?

Poisson inspection, ctd.

Proposition

Assume $B \in \mathscr{L}$. As $u \to \infty$,

$$\frac{\bar{p}(u)}{p(u)} \to \gamma^\star_\omega := \frac{\psi(\omega)}{\psi(\omega) + \theta^\star}.$$

Proof: see book. Main idea: net cumulative claim process $Y^\circ(t)$ (i.e., different from our actual net cumulative claim process $Y(t)$, viz. with claims Z^+, exponentially distributed interclaim times Z^-, and unit premium rate) exceeds u, and then use result from Section 2.2.

This shows: $\gamma^\star_\omega \uparrow 1$ as ω grows large, as expected.

Length of first excursion

As before: $\tau(u)$ ruin time.

In addition, $U^\circ(u)$: length of interval after $\tau(u)$ at which level $X_u(t)$ uninterruptedly attains a negative value (or: net cumulative claim process $Y(t)$ uninterruptedly attains a value above u).

Then,

$$V_\beta(u) := \min\{U^\circ(u), T_\beta - \tau(u)\}\, 1\{\tau(u) < T_\beta\}.$$

Of interest when bankruptcy occurs when length of first excursion exceeds some threshold.

Length of first excursion, ctd.

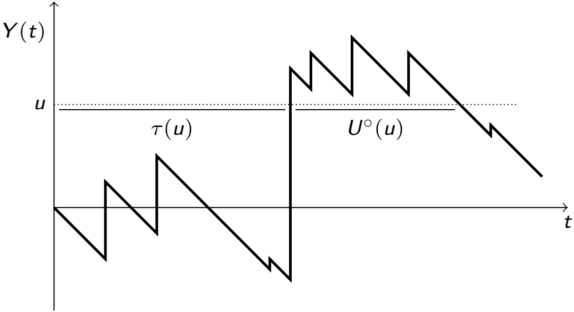

Figure: Net cumulative claim process $Y(t)$, and quantities $\tau(u)$ and $U^{\circ}(u)$.

Length of first excursion, ctd.

We know from Section 5.4 how to compute overshoot (through its transform)

$$\mathbb{P}(Y(\tau(u)) - u \in dy, \tau(u) \leqslant T_\beta);$$

corresponding density is called $h(u, y, \beta)$.

Hence, by memoryless property of exponential distribution,

$$\mathbb{E}\, e^{-\alpha V_\beta(u)} = \int_0^\infty h(u, y, \beta)\, \mathbb{E}\, e^{-\alpha \min\{\sigma(y), T_\beta\}}\, dy,$$

with $\sigma(u)$ time it takes for $Y(t)$ to decrease by at least u.

Lemma 1.1: for any $y > 0$,

$$\mathbb{E}\, e^{-\alpha\sigma(y)} = e^{-\psi(\alpha)\, y}.$$

Length of first excursion, ctd.

Lemma

For any $\alpha \geqslant 0$ and $\beta > 0$, and for any non-negative random variable X that is independent of T_β,

$$\mathbb{E}\, e^{-\alpha \min\{X, T_\beta\}} = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}\, \mathbb{E}\, e^{-(\alpha+\beta)X}.$$

Proof. Rewrite $\mathbb{E}\, e^{-\alpha \min\{X, T_\beta\}}$ by conditioning on T_β:

$$\int_0^\infty \beta e^{-\beta t}\, \mathbb{E}\, e^{-\alpha \min\{X,t\}}\, dt$$
$$= \int_0^\infty \beta e^{-\beta t} \int_0^t \mathbb{P}(X \in dx)\, e^{-\alpha x}\, dt + \int_0^\infty \beta e^{-\beta t} \int_t^\infty \mathbb{P}(X \in dx)\, e^{-\alpha t}\, dt.$$

Then: swap order of integrals, evaluate integrals over t, and interpret the obtained expressions in terms of the LST of X.

Length of first excursion, ctd.

Combining the above (including the use of Lemma),

$$\begin{aligned}\mathbb{E}\, e^{-\alpha V_\beta(u)} &= \int_0^\infty h(u,y,\beta)\left(\frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}\mathbb{E}\, e^{-(\alpha+\beta)\sigma(y)}\right) dy \\ &= \frac{\beta}{\alpha+\beta}\, p(u,T_\beta) + \frac{\alpha}{\alpha+\beta}\int_0^\infty h(u,y,\beta)\, e^{-\psi(\alpha+\beta)\, y}\, dy.\end{aligned}$$

Interpret integral in terms of overshoot $Y(\tau(u)) - u$. Define

$$\chi(u,\alpha,\beta) := \mathbb{E}\Big(e^{-\psi(\alpha+\beta)\,(Y(\tau(u))-u)}\, 1\{\tau(u) \leqslant T_\beta\}\Big).$$

Proposition

For any $\alpha \geqslant 0$ and $\beta > 0$,

$$\mathbb{E}\, e^{-\alpha V_\beta(u)} = \frac{\beta}{\alpha+\beta}\, p(u,T_\beta) + \frac{\alpha}{\alpha+\beta}\, \chi(u,\alpha,\beta).$$

Length of first excursion, ctd.

All expressions appearing in Proposition can be assessed. Indeed,

- as computed in Section 1.3,

$$\int_0^\infty e^{-\omega u}\, p(u, T_\beta)\, du = \pi(\omega, \beta) = \frac{1}{\varphi(\omega) - \beta}\left(\frac{\varphi(\omega)}{\omega} - \frac{\beta}{\psi(\beta)}\right),$$

- whereas the transform of $\chi(u, \alpha, \beta)$ follows from

$$\begin{aligned}\kappa(\omega, \beta, \gamma) &:= \int_0^\infty e^{-\omega u}\, \mathbb{E}(e^{-\gamma\,(Y(\tau(u)) - u)} 1\{\tau(u) \leqslant T_\beta\})\, du \\ &= \frac{\lambda}{\varphi(\omega) - \beta}\left(\frac{b(\psi(\beta)) - b(\gamma)}{\gamma - \psi(\beta)} - \frac{b(\omega) - b(\gamma)}{\gamma - \omega}\right),\end{aligned}$$

  using the analysis of Sections 5.3–5.4, leading to

$$\int_0^\infty e^{-\omega u}\, \chi(u, \alpha, \beta)\, du = \kappa(\omega, \beta, \psi(\alpha + \beta)).$$

Total time with negative surplus

Consider total time (until exponential killing) that net cumulative claim process is larger than u:

$$W_\beta(u) := \int_0^{T_\beta} 1\{X_u(t) \leqslant 0\}\, dt = \int_0^{T_\beta} 1\{Y(t) \geqslant u\}\, dt.$$

Is of importance when bankruptcy depends on time surplus level is below zero.

We analyze $W_\beta(u)$ through its transform (with respect to u). Three disjoint events:
(i) $\{\tau(u) + U^\circ(u) \leqslant T_\beta\}$,
(ii) $\{\tau(u) \leqslant T_\beta < \tau(u) + U^\circ(u)\}$, and
(iii) $\{T_\beta < \tau(u)\}$.

Total time with negative surplus, ctd.

- Case (i) gives contribution

$$\mathbb{E}\left(e^{-\alpha U^\circ(u)} 1\{\tau(u) + U^\circ(u) < T_\beta\}\right) \mathbb{E}\, e^{-\alpha W_\beta(0)},$$

  which equals $\chi(u,\alpha,\beta)\,\mathbb{E}\, e^{-\alpha W_\beta(0)}$ (Exercise 10.2).

- Calling $\tilde{T}_\beta$ remaining part of T_β given that $T_\beta \geqslant \tau(u)$ plus memoryless property: the contribution of Case (ii) is

$$\begin{aligned}
&\mathbb{E}\left(e^{-\alpha \tilde{T}_\beta} 1\{\tau(u) \leqslant T_\beta, U^\circ(u) > \tilde{T}_\beta\}\right) \\
&\quad = \mathbb{E}\left(e^{-\alpha \tilde{T}_\beta} 1\{\tau(u) \leqslant T_\beta\}\right) - \mathbb{E}\left(e^{-\alpha \tilde{T}_\beta} 1\{\tau(u) \leqslant T_\beta, U^\circ(u) \leqslant \tilde{T}_\beta\}\right) \\
&\quad = \frac{\beta}{\beta+\alpha}\, p(u, T_\beta) - \frac{\beta}{\beta+\alpha}\, \chi(u,\alpha,\beta).
\end{aligned}$$

- Case (iii) finally contributes $\mathbb{P}(T_\beta < \tau(u)) = 1 - p(u, T_\beta)$.

Total time with negative surplus, ctd.

Adding the three contributions:

$$\mathbb{E}\, e^{-\alpha W_\beta(u)} = \chi(u,\alpha,\beta)\,\mathbb{E}\, e^{-\alpha W_\beta(0)} + 1 - \frac{\alpha}{\alpha+\beta}\, p(u, T_\beta) - \frac{\beta}{\alpha+\beta}\,\chi(u,\alpha,\beta).$$

Recall: we can evaluate the transform (to u) of $p(u, T_\beta)$ and $\chi(u,\alpha,\beta)$

Hence, we are left with analyzing $\mathbb{E}\, e^{-\alpha W_\beta(0)}$.

Total time with negative surplus, ctd.

To compute $\mathbb{E}\, e^{-\alpha W_\beta(0)}$, we work with two auxiliary random sequences:

- Let D_i be the length of the i-th uninterrupted period that $Y(t)$ is negative ('down');
- likewise, we let U_i be the length of the i-th uninterrupted period that $Y(t)$ is non-negative ('up').

Observe: $(D_i, U_i)_{i\in\mathbb{N}}$ is sequence of i.i.d. two-dimensional random vectors; let (D, U) denote corresponding generic random vector.

Exploiting the regenerative structure,

$$\begin{aligned}\mathbb{E}\, e^{-\alpha W_\beta(0)} = {} & \mathbb{E}\big(e^{-\alpha U} 1\{D + U \leqslant T_\beta\}\big)\, \mathbb{E}\, e^{-\alpha W_\beta(0)} + \\ & \mathbb{E}\big(e^{-\alpha(T_\beta - D)} 1\{D \leqslant T_\beta < D + U\}\big) + \mathbb{P}(T_\beta < D).\end{aligned}$$

Goal: evaluate three unknown quantities.

Total time with negative surplus, ctd.

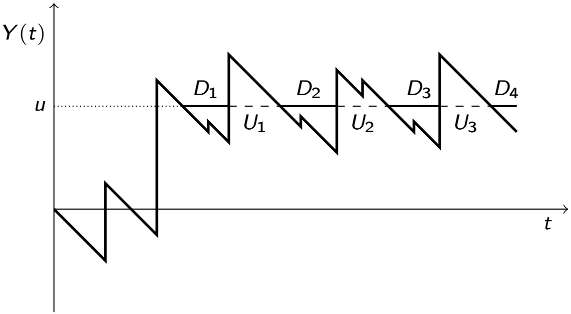

Figure: Net cumulative claim process $Y(t)$, and the quantities $(D_i, U_i)_{i\in\mathbb{N}}$.

Total time with negative surplus, ctd.

Start with $\Omega_1(\beta) := \mathbb{P}(T_\beta < D)$.

$\mathbb{P}(T_\beta \geqslant D)$ can be rewritten as

$$\int_0^\infty \lambda e^{-(\lambda+\beta)t} \left(\int_0^{rt} \mathbb{P}(B \in dv)\, \mathbb{P}(T_\beta > \tau(rt-v)) + \int_{rt}^\infty \mathbb{P}(B \in dv) \right) dt$$

by conditioning on the first claim arrival time. Recalling that $\mathbb{P}(T_\beta > \tau(u)) = p(u, T_\beta)$, performing the change of variable $s = rt$, and splitting the exponent, this expression equals

$$\frac{\lambda}{r} \int_0^\infty \int_0^s \mathbb{P}(B \in dv)\, e^{-(\lambda+\beta)\, v/r}\, p(s-v, T_\beta)\, e^{-(\lambda+\beta)\,(s-v)/r}\, ds +$$
$$\frac{\lambda}{r} \int_0^\infty \int_s^\infty \mathbb{P}(B \in dv)\, e^{-(\lambda+\beta)\, s/r}\, ds.$$

Total time with negative surplus, ctd.

Evaluating integrals in standard way, and recognizing underlying convolution structure,

$$\Omega_1(\beta) = \frac{\beta}{\lambda+\beta} + \frac{\lambda}{\lambda+\beta}\, b\left(\frac{\lambda+\beta}{r}\right) - \frac{\lambda}{r}\, b\left(\frac{\lambda+\beta}{r}\right)\pi\left(\frac{\lambda+\beta}{r}, \beta\right),$$

with $\pi(\alpha, \beta)$ as given in Section 1.3. After some calculus:

$$\Omega_1(\beta) = \frac{\beta}{r\psi(\beta)}.$$

Total time with negative surplus, ctd.

We now focus on $\Omega_2(\alpha,\beta) := \mathbb{E}(e^{-\alpha U} 1\{D+U \leqslant T_\beta\})$. For conciseness: Y_u^+ the overshoot over level u, i.e., $Y(\tau(u)) - u$.

Then $\Omega_2(\alpha,\beta)$ equals, again by conditioning on first claim arrival time,

$$\int_0^\infty \lambda e^{-(\lambda+\beta)t} \Bigg(\int_0^{rt} \mathbb{P}(B \in dv) \int_{rt-v}^\infty \mathbb{P}(Y_{rt-v}^+ \in dy, \tau(rt-v) \leqslant T_\beta) \\ \mathbb{E}(e^{-\alpha\sigma(y)} 1\{\sigma(y) \leqslant T_\beta\}) \\ + \int_{rt}^\infty \mathbb{P}(B \in dv)\, \mathbb{E}(e^{-\alpha\sigma(v-rt)} 1\{\sigma(v-rt) \leqslant T_\beta\}) \Bigg) dt;$$

distinguish between (i) scenario in which after first claim arrival (before T_β) net cumulative claim process is below 0 (first term between brackets), and (ii) scenario in which at first claim arrival net cumulative claim process has exceeded 0 (second term between brackets). Expression has been set up such that at first claim arrival (at end of D) and at the end of U, killing time T_β has not expired.

Total time with negative surplus, ctd.

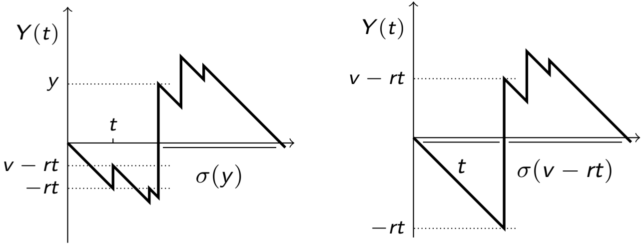

Figure: Net cumulative claim process $Y(t)$ in scenario that multiple claims are needed to exceed 0 (left panel), and in scenario that one claim suffices (right panel). Left panel: first jump (of size $v < rt$) happens at time t, eventually leading to overshoot of y over level 0. Right panel first jump (of size $v \geqslant rt$) happens at time t, directly leading to overshoot of $v - rt$ over level 0.

Total time with negative surplus, ctd.

Observe that, for any $\alpha \geqslant 0$ and $\beta > 0$, and $y > 0$,

$$\mathbb{E}(e^{-\alpha\sigma(y)}\, 1\{\sigma(y) \leqslant T_\beta\}) = \int_0^\infty \int_0^t e^{-\alpha x} \beta e^{-\beta t}\, \mathbb{P}(\sigma(y) \in dx)\, dt$$
$$= \int_0^\infty e^{-(\alpha+\beta)\,x}\, \mathbb{P}(\sigma(y) \in dx)$$
$$= \mathbb{E}\, e^{-(\alpha+\beta)\,\sigma(y)} = e^{-\psi(\alpha+\beta)\,y}.$$

Hence, substituting s for rt, $\Omega_2(\alpha, \beta)$ is sum of two terms:

$$\frac{\lambda}{r} \int_0^\infty e^{-(\lambda+\beta)\,s/r} \int_0^s \mathbb{P}(B \in dv)\, \mathbb{E}\Big(e^{-\psi(\alpha+\beta)\,Y^+_{s-v}}\, 1\{\tau(s-v) \leqslant T_\beta\}\Big)\, ds,$$

and

$$\frac{\lambda}{r} \int_0^\infty e^{-(\lambda+\beta)\,s/r} \int_s^\infty \mathbb{P}(B \in dv)\, e^{-\psi(\alpha+\beta)\,(v-s)}\, ds.$$

Total time with negative surplus, ctd.

Recognizing convolution structure, first term is

$$\frac{\lambda}{r}\, b\left(\frac{\lambda+\beta}{r}\right)\,\kappa\left(\frac{\lambda+\beta}{r},\beta,\psi(\alpha+\beta)\right).$$

Second term: swap order of integrals (and elementary calculus):

$$\lambda\,\frac{b(\psi(\alpha+\beta))-b((\lambda+\beta)/r)}{\lambda+\beta-r\psi(\alpha+\beta)}.$$

We conclude that

$$\begin{aligned}\Omega_2(\alpha,\beta) &= \frac{\lambda}{r}\, b\left(\frac{\lambda+\beta}{r}\right)\,\kappa\left(\frac{\lambda+\beta}{r},\beta,\psi(\alpha+\beta)\right) + \\ &\qquad \lambda\,\frac{b(\psi(\alpha+\beta))-b((\lambda+\beta)/r)}{\lambda+\beta-r\psi(\alpha+\beta)}.\end{aligned}$$

Inserting (known) expression for $\kappa(\alpha,\beta,\gamma)$, we eventually find

$$\Omega_2(\alpha,\beta)=\frac{\lambda}{r}\,\frac{b(\psi(\beta))-b(\psi(\alpha+\beta))}{\psi(\alpha+\beta)-\psi(\beta)}.$$

Total time with negative surplus, ctd.

Then: $\Omega_3(\alpha,\beta) := \mathbb{E}(e^{-\alpha(T_\beta - D)}1\{D \leqslant T_\beta < D+U\})$. Again by conditioning on first claim arrival time,

$$\int_0^\infty \lambda e^{-(\lambda+\beta)t}\left(\int_0^{rt}\mathbb{P}(B\in dv)\int_{rt-v}^\infty \mathbb{P}(Y^+_{rt-v}\in dy, \tau(rt-v)\leqslant T_\beta)\right.$$
$$\mathbb{E}(e^{-\alpha T_\beta}1\{\sigma(y) > T_\beta\})$$
$$\left. + \int_{rt}^\infty \mathbb{P}(B\in dv)\,\mathbb{E}(e^{-\alpha T_\beta}1\{\sigma(v-rt) > T_\beta\})\right)dt.$$

For any $\alpha \geqslant 0$ and $\beta > 0$, and $y > 0$,

$$\begin{aligned}\mathbb{E}(e^{-\alpha T_\beta}\,1\{\sigma(y) > T_\beta\}) &= \int_0^\infty e^{-\alpha t}\beta e^{-\beta t}\,\mathbb{P}(\sigma(y) > t)\,dt\\ &= \frac{\beta}{\alpha+\beta}\,\mathbb{P}(\sigma(y) > T_{\alpha+\beta})\\ &= \frac{\beta}{\alpha+\beta}\left(1 - e^{-\psi(\alpha+\beta)\,y}\right).\end{aligned}$$

Total time with negative surplus, ctd.

Using same techniques as before,

$$\begin{aligned}\Omega_3(\alpha,\beta) &= \frac{\lambda}{r}\frac{\beta}{\alpha+\beta}\,b\left(\frac{\lambda+\beta}{r}\right)\cdot\left(\kappa\left(\frac{\lambda+\beta}{r},\beta,0\right)-\kappa\left(\frac{\lambda+\beta}{r},\beta,\psi(\alpha+\beta)\right)\right)\\ &+\frac{\lambda\beta}{\alpha+\beta}\left(\frac{1-b((\lambda+\beta)/r)}{\lambda+\beta}-\frac{b(\psi(\alpha+\beta))-b((\lambda+\beta)/r)}{\lambda+\beta-r\psi(\alpha+\beta)}\right).\end{aligned}$$

Considerable calculus: simplifies to

$$\Omega_3(\alpha,\beta)=\frac{\lambda}{r}\frac{\beta}{\alpha+\beta}\left(\frac{1-b(\psi(\beta))}{\psi(\beta)}-\frac{b(\psi(\beta))-b(\psi(\alpha+\beta))}{\psi(\alpha+\beta)-\psi(\beta)}\right).$$

Total time with negative surplus, ctd.

Upon collecting above results, we have identified transform of $W_\beta(u)$.

Theorem

For any $\alpha \geqslant 0$ and $\beta > 0$, $\mathbb{E}\, e^{-\alpha W_\beta(u)}$ is given by

$$\mathbb{E}\, e^{-\alpha W_\beta(u)} = \chi(u,\alpha,\beta)\,\mathbb{E}\, e^{-\alpha W_\beta(0)} + 1 - \frac{\alpha}{\alpha+\beta}\, p(u, T_\beta) - \frac{\beta}{\alpha+\beta}\,\chi(u,\alpha,\beta)$$

with transforms of $p(u, T_\beta)$ and $\chi(u,\alpha,\beta)$ as given above, and

$$\mathbb{E}\, e^{-\alpha W_\beta(0)} = \frac{\Omega_1(\beta) + \Omega_3(\alpha,\beta)}{1 - \Omega_2(\alpha,\beta)}.$$