Steps Towards A Yield-Macro Model for Actuarial Use

Şule Şahin

Prof Andrew Cairns, Dr Torsten Kleinow, Prof David Wilkie

Heriot-Watt University, Edinburgh/UK

Hacettepe University, Ankara/Turkey

27 March 2009
Outline

1. Wilkie Investment Model
2. Term structure of interest rates (Yield curve)
3. A Descriptive yield curve model: Cairns model
   ▶ Cairns Model
   ▶ Deciding the best set of 'exponential rates' of the Cairns Model
     ▶ Standard errors
     ▶ Observed and fitted values for specific dates
4. Principal component analysis (PCA)
   ▶ Principal components of the nominal interest rates
   ▶ Principal components of the implied inflation rates
5. Future Works: Constructing a yield-macro model
Wilkie Investment Model

Figure: Structure of the Wilkie model

Retail Price Index (Inflation) → Share Dividend Yields → Share Dividends, Share Prices

Share Dividend Yields → Long-Term Interest Rates → Short-Term Interest Rates
Revisiting the Wilkie Investment Model

- Retail prices index models and long-term interest rate models do not satisfy the normality assumption.
- No significant change in the performance of the models for share dividends, share dividend yields and short-term interest rates.
- Most of the parameters have not been stable.
Term Structure of Interest Rates (Yield Curve)

- **Bond:** A securitized form of loan in which the *issuer* is the borrower, the *bondholder* is the lender, and the *coupon* is the interest.

- **Zero-coupon bond:** An n-year zero-coupon bond is a security which pays 1 after n years and nothing else, i.e. a fixed-interest security, redeemed at par, with term n years and no coupon payments.

- **Yield curve:** The relation between the *interest rate* (or cost of borrowing) and the *time to maturity* of the debt.
Term Structure of Interest Rates (Yield Curve)

- **Inflation**: A rise in the general level of prices of goods and services in an economy over a period of time.
  - **Realised inflation**: The actual inflation experienced over one month in the recent past.
  - **Implied inflation**: Tied to financial products which reference future realised inflation.

- **GDP (Output)**: Value of a country’s overall output of goods and services at market prices.
Term Structure of Interest Rates (Yield Curve)
Term Structure of Interest Rates (Yield Curve)
Term Structure of Interest Rates (Yield Curve)
Term Structure of Interest Rates (Yield Curve)
A Descriptive Yield Curve Model: Cairns Model

A **descriptive model** takes a snapshot of the bond market as it is today. The sole aim is to get a good description of todays prices: that is, of the rates of interest which are implicit in todays prices.

- Construction of yield indices


- 7457 observations (daily nominal interest rates) which are based on half year maturities (6 months to 25 years i.e. 50 different maturities).
Cairns Model

- \( f_{kT} \): spot rate
- \( k \): observation date (\( k = 1, 2, \ldots, 7457 \) )
- \( T \): maturity (\( T = 0.5, 1, \ldots, 25 \) )

\[
\hat{f}_{kT} = b_0(t) + b_1(t) \frac{1 - e^{-c_1 T}}{c_1 T} + b_2(t) \frac{1 - e^{-c_2 T}}{c_2 T} + b_3(t) \frac{1 - e^{-c_3 T}}{c_3 T} + b_4(t) \frac{1 - e^{-c_4 T}}{c_4 T}
\]

- C Parameter Sets \( C = (c_1, c_2, c_3, c_4) \)

\[
C_1 = (0.2, 0.4, 0.8, 1.6), \quad C_2 = (0.1, 0.2, 0.3, 0.8) \quad C_3 = (0.2, 0.4, 0.6, 0.8), \quad COpt = (0.05, 0.10, 0.26, 0.87)
\]
Factor Loadings for $b$ Parameters

$C_1 = (0.2, 0.4, 0.8, 1.6)$

$C_2 = (0.1, 0.2, 0.4, 0.8)$

$C_3 = (0.2, 0.4, 0.6, 0.8)$

$C_{opt} = (0.05, 0.10, 0.26, 0.87)$
Mean Square Errors (MSE)

\[
\text{Mean Square Error (MSE)}^2_k = \frac{\sum_{t=1}^{T}(\hat{f}_{kt} - f_{kt})^2}{T}
\]

\[
\text{Root Mean Square Error (RMSE)} = \sqrt{\text{MSE}^2_k}
\]

where \( T = 1, \ldots, 50 \) is the associated maturity of the observed day \( k = 1, \ldots, 7457 \).
RMSE (Overall Period)
Ratios of RMSE (Overall Period)

RMSE2/RMSE1

RMSE3/RMSE1

RMSEOpt/RMSE1

Steps Towards A Yield-Macro Model for Actuarial Use
Ratios of RMSE (Short-Term)
Ratios of RMSE (Medium-Term)
Ratios of RMSE (Long-Term)

Steps Towards A Yield-Macro Model for Actuarial Use
Standard Error Analysis

The mean RMSEs calculated as below:

\[
\text{Mean RMSE} = \frac{\sum_{k=1}^{7457} \text{RMSE}_k}{7457}
\]

<table>
<thead>
<tr>
<th></th>
<th>Overall period</th>
<th>Short term</th>
<th>Medium term</th>
<th>Long term</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 1</td>
<td>3.5833</td>
<td>3.9838</td>
<td>3.2049</td>
<td>3.9564</td>
</tr>
<tr>
<td>Model 2</td>
<td><strong>2.2903</strong></td>
<td>3.6332</td>
<td><strong>1.6009</strong></td>
<td><strong>1.6328</strong></td>
</tr>
<tr>
<td>Model 3</td>
<td>2.5306</td>
<td><strong>3.3118</strong></td>
<td>2.0034</td>
<td>2.5729</td>
</tr>
<tr>
<td>Model 4</td>
<td>2.5891</td>
<td>3.9121</td>
<td>1.9604</td>
<td>1.9932</td>
</tr>
</tbody>
</table>

Table: Mean RMSE (bps) for Different C Parameter Sets
Observed and Fitted Values (1979-01-02)

C1=(0.2,0.4,0.8,1.6)

C2=(0.1,0.2,0.4,0.8)

C3=(0.2,0.4,0.6,0.8)

COpt=(0.05, 0.10, 0.26, 0.87)

Şule Şahin
Steps Towards A Yield-Macro Model for Actuarial Use
Observed and Fitted Values (1982-12-13)

C1=(0.2,0.4,0.8,1.6)

C2=(0.1,0.2,0.4,0.8)

C3=(0.2,0.4,0.6,0.8)

COpt=(0.05 0.10 0.26 0.87)

Süleyman Şahin
Steps Towards A Yield-Macro Model for Actuarial Use
Observed and Fitted Values (2002-09-25)

C1=(0.2,0.4,0.8,1.6)

C2=(0.1,0.2,0.4,0.8)

C3=(0.2,0.4,0.6,0.8)

COpt=(0.05 0.10 0.26 0.87)

Süle Şahin
Steps Towards A Yield-Macro Model for Actuarial Use
Observed and Fitted Values (2008-05-22)

C1=(0.2,0.4,0.8,1.6)

C2=(0.1,0.2,0.4,0.8)

C3=(0.2,0.4,0.6,0.8)

COpt=(0.05,0.10,0.26,0.87)

Süle Şahin
Steps Towards A Yield-Macro Model for Actuarial Use
Principal Component Analysis (PCA)

**Aim:** To reduce the dimensionality of a data set which consists of a large number of interrelated variables, while retaining as much as possible of the variation present in the data set.

- Achieved by transforming to a new set of variables (principal components)
- Uncorrelated
- Ordered
- Eigenvalue-eigenvector problem for a positive-semidefinite symmetric matrix (Jolliffe, 1986)
Principal Component Analysis (PCA)

Suppose \( x \) is a vector of \( p \) random variables and \( \alpha_1 \) is a vector of \( p \) constants, \( \alpha_{11}, \alpha_{12}, \ldots, \alpha_{1p} \).

**Step 1:** Look for a linear function \( \alpha'_1 x \) of the elements of \( x \) which has a maximum variance so that

\[
\alpha'_1 x = \alpha_{11}x_1 + \alpha_{12}x_2 + \ldots + \alpha_{1p}x_p = \sum_{j=1}^{p} \alpha_{1j}x_j
\]

**Step 2:** Look for a linear function \( \alpha'_2 x \), uncorrelated with \( \alpha'_1 x \) which has maximum variance

**Step k:** Look for a linear function \( \alpha'_k x \) which has maximum variance subject to being uncorrelated with \( \alpha'_1 x, \alpha'_2 x, \ldots, \alpha'_k x \)

The \( k \)th derived variable, \( z_k = \alpha'_k x \), is the \( k \)th PC.
First Principal Component

Consider first $\alpha_1' x$; $\alpha_1$ maximizes

$$\text{var} [\alpha_1' x] = \alpha_1' \Sigma \alpha_1$$

A normalization constraint must be imposed: $\alpha_1' \mathbf{1} = 1$

To maximize $\alpha_1' \Sigma \alpha_1$ subject to $\alpha_1' \mathbf{1} = 1$,

$$\alpha_1' \Sigma \alpha_1 - \lambda (\alpha_1' \alpha_1 - 1)$$

where $\lambda$ is a Lagrange multiplier. Differentiation with respect to $\alpha_1$ gives

$$\Sigma \alpha_1 - \lambda \mathbf{1} \alpha_1 = 0$$

or

$$(\Sigma - \lambda \mathbf{I}_p) \alpha_1 = 0$$

where $\mathbf{I}_p$ is the $(p \times p)$ identity matrix. Thus, $\lambda$ is an eigenvalue of $\Sigma$ and $\alpha_1$ is the corresponding eigenvector.
PCs of the Yield Changes


- $\mathbf{x}$ is $n \times p$ matrix of *nominal spot rates* where $n = 1, \ldots, 289$ and $p = 1, \ldots, 50$.

- $\mathbf{y}$ is $n \times k$ matrix of *implied inflation spot rates* where $n = 1, \ldots, 289$ and $k = 1, \ldots, 46$.

- Take the first differences (*yield changes*) and calculate the eigenvalues and eigenvectors of the correlation matrices of $\mathbf{x}$ and $\mathbf{y}$.

- First three principal components explain more than 99% of the variability in both data.
Eigenvalues (Loadings of the PCs) of the Yield Changes

Nominal Interest Rates

Implied Inflation Rates

Variances

Variance
Loadings of the PCs

PCs of Nominal Interest Rates

PCs of Implied Inflation Rates

Steps Towards A Yield-Macro Model for Actuarial Use
 PCs of the Nominal and Implied Inflation Yield Changes

- **PCN**: Principal components of the *nominal* spot yield changes
- **PCI**: Principal components of the *implied inflation* spot yield changes
- $\alpha_N$ and $\alpha_I$ are the eigenvectors of the correlation matrices of the nominal and implied inflation yield changes

\[
\text{PCN}_{[288 \times 3]} = \mathbf{x}_{[288 \times 50]} \times \alpha_{N}[50 \times 3]
\]
\[
\text{PCI}_{[288 \times 3]} = \mathbf{y}_{[288 \times 46]} \times \alpha_{I}[46 \times 3]
\]
PCs of the Nominal Yield Changes

Steps Towards A Yield-Macro Model for Actuarial Use
PCs of the Implied Inflation Yield Changes

Steps Towards A Yield-Macro Model for Actuarial Use

Süle Şahin
Correlations between the PCs of the Nominal and Implied Inflation Yield Changes

Süle Şahin
Steps Towards A Yield-Macro Model for Actuarial Use
Future Works: Constructing a Yield-Macro Model

- Bridging the gap
  - between the yield-curve models developed by the macroeconomists and financial economists by studying
    - Macroeconomic interpretations for the ‘level’, ‘slope’ and ‘curvature’ (long-run inflation expectations, monetary policy, output gap)
    - Link between the term structure of implied inflation and the term structure of nominal interest rates
    - Link between the realized inflation and the term structure of implied inflation
- Methodology
  - VAR, ARCH, GARCH models

Şule Şahin
Steps Towards A Yield-Macro Model for Actuarial Use
References